

DISCRIMINATION OF LARGE QUANTUM ENSEMBLES

Emilio Bagan



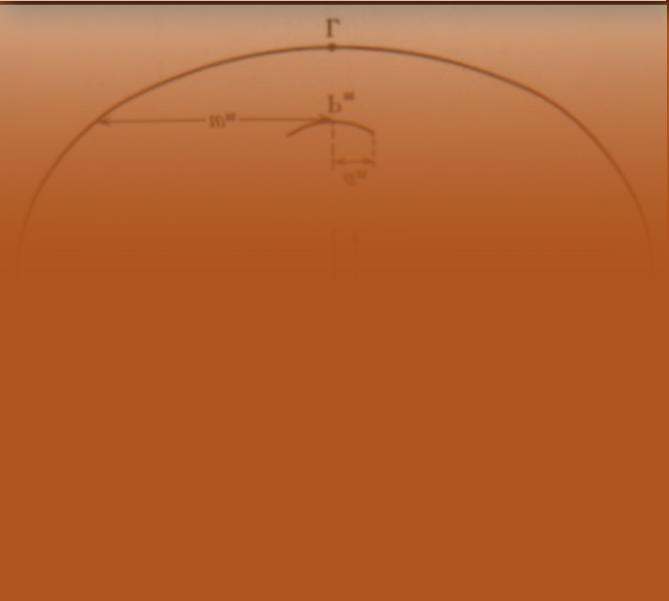
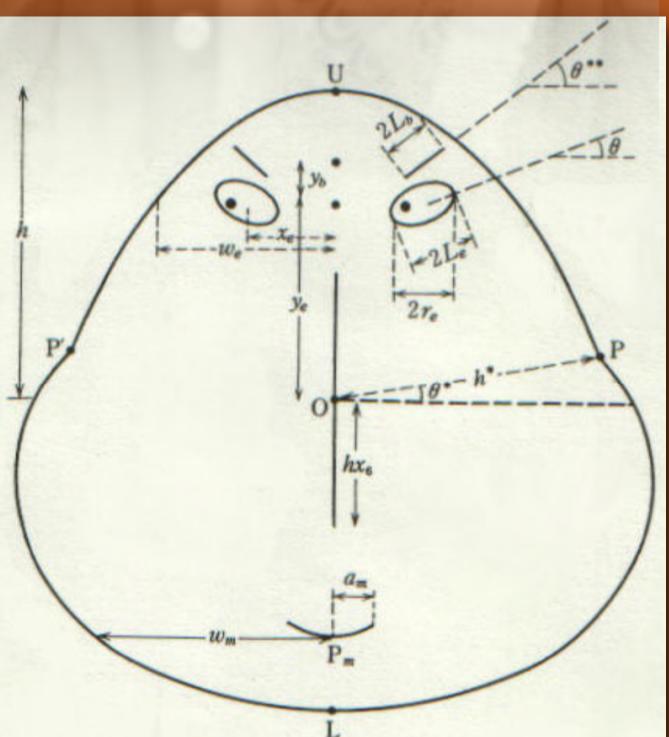
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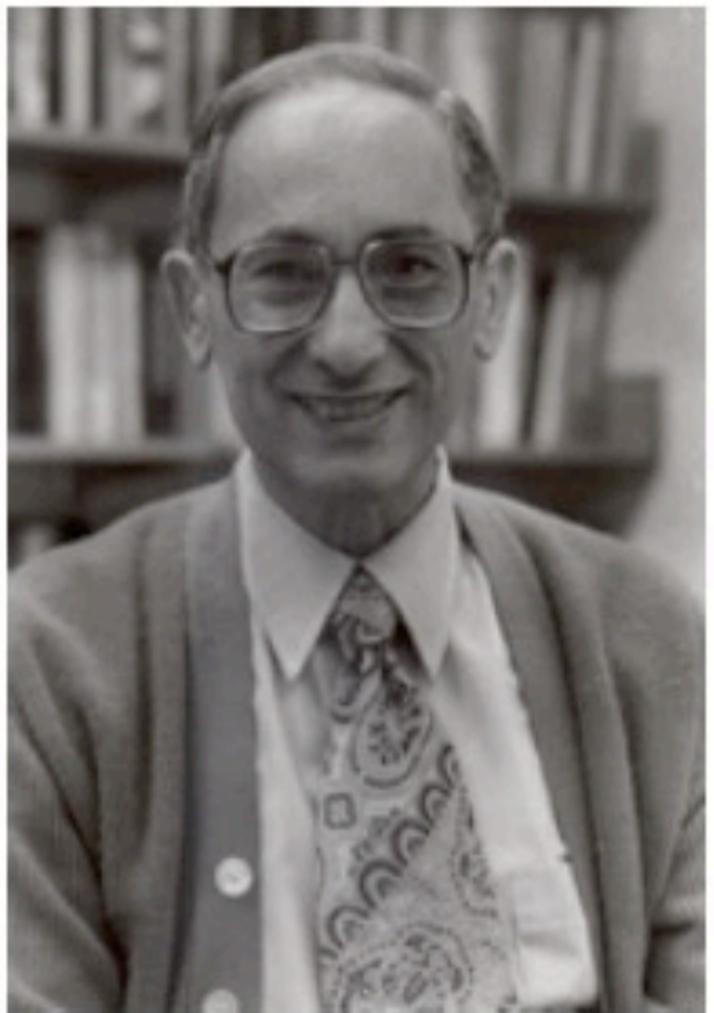
K.M.R. Audenaert, J Calsamiglia, LI. Masanes, R. Munoz-Tapia, A. Acin
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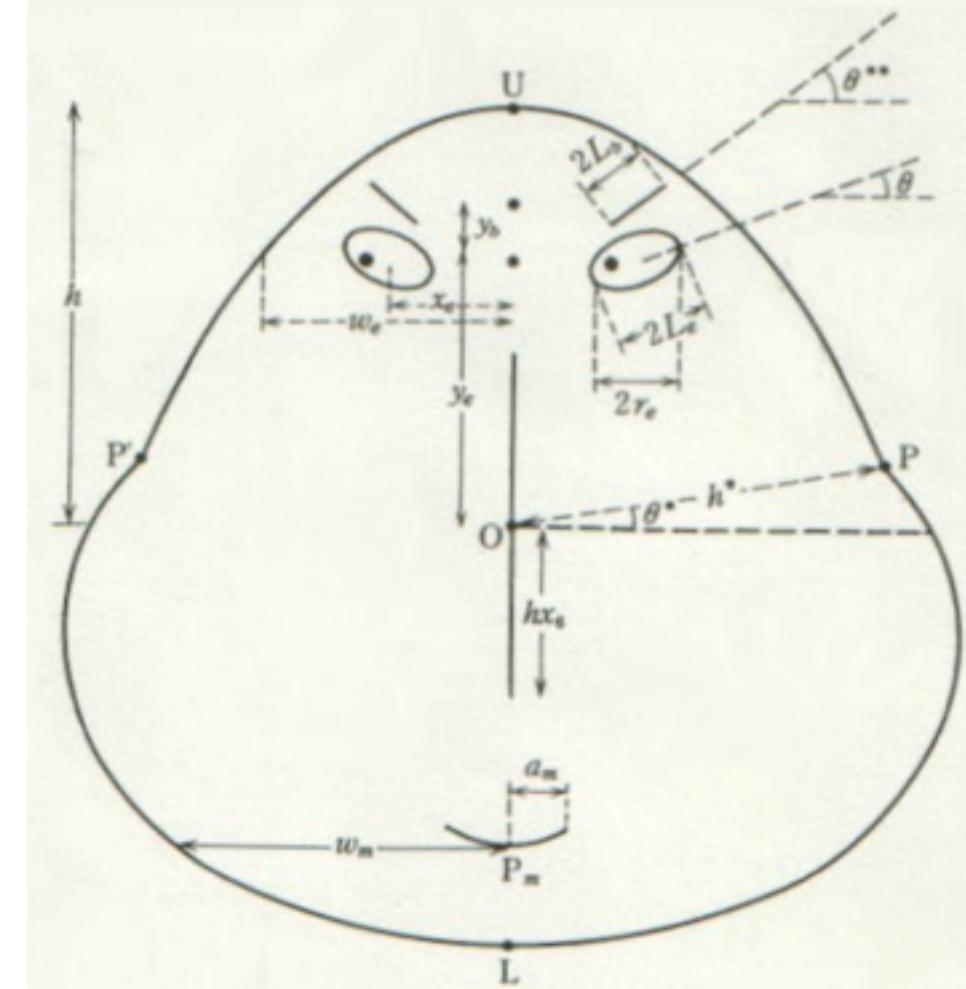
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Professor Chernoff is Professor Emeritus of Applied Mathematics at MIT and of Statistics at Harvard University. His major fields of research have been in large Sample Theory, Optimal Design of Experiments, Sequential Analysis, and Sequential Design of Experiments. In recent years he has worked on a series of statistical issues in Molecular Biology.

- Bounds on tails of probability distributions
- Bound on averaged error probability in symmetric hypothesis testing

- Chernoff faces: Visualization technique to illustrate trends in multidimensional data

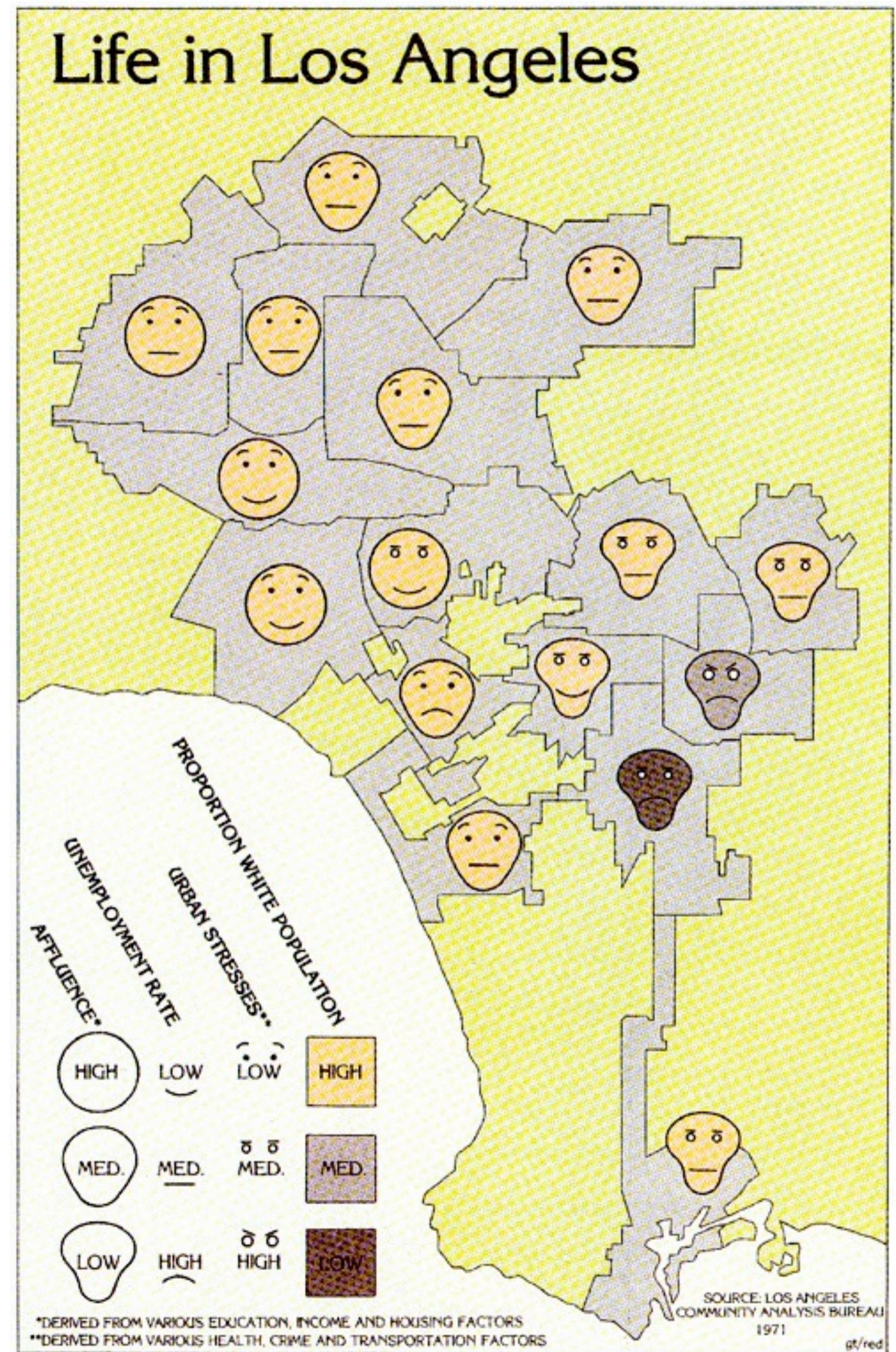


Chernoff bounds

If $X \sim \text{Binomial}(n, p)$

$$\begin{aligned}\mathbb{P}(X - np \geq \delta) &\leq e^{-2\delta^2/n} \\ \mathbb{P}(X - np \leq -\delta) &\leq e^{-2\delta^2/n}\end{aligned}$$

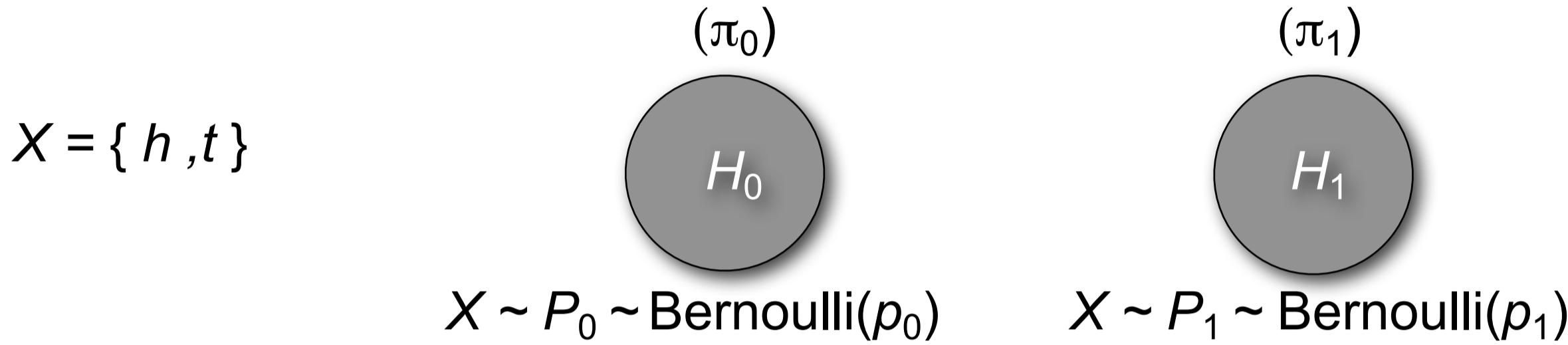
Chernoff faces



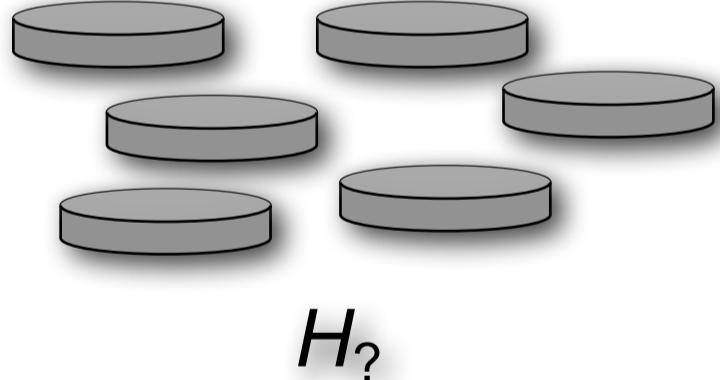
CLASSICAL

Hypothesis testing/discrimination

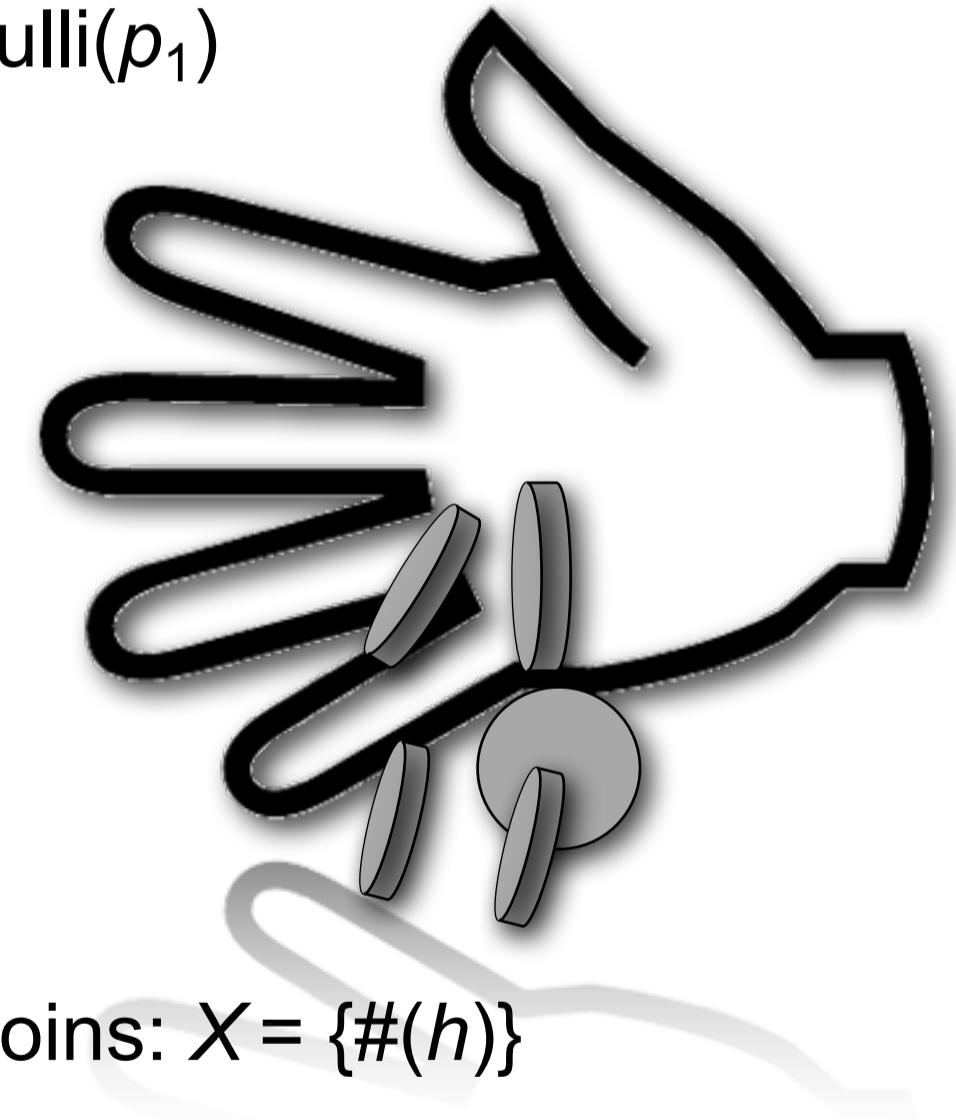
Simple example: consider two different kind of coins



We are given N identical coins



$H_?$



Toss the coins: $X = \{\#(h)\}$

Decide if we have been given H_0 or H_1 with minimal averaged error probability

$$P_e = \pi_0 \text{ prob}(H_? = H_1 | H_0) + \pi_1 \text{ prob}(H_? = H_0 | H_1)$$

P_e Fall off exponentially with N ; exponential rate: Chernoff bound

CLASSICAL

Hypothesis testing/discrimination

Two-coin discrimination is just the simplest instance of a vast field

Hypothesis testing is a powerful tool of broad use in science, economics, engineering,...

- H_0 Null hypothesis
- H_1 Alternative hypothesis
- Error probabilities (risk):
 - Type I: $\alpha = \text{prob}(\text{reject } H_0 \text{ when it is true})$
 - Type II: $\beta = \text{prob}(\text{accept } H_0 \text{ when it is false})$
- Symmetric *Bayesian* setting: $P_e = \pi_0 \alpha + \pi_1 \beta$

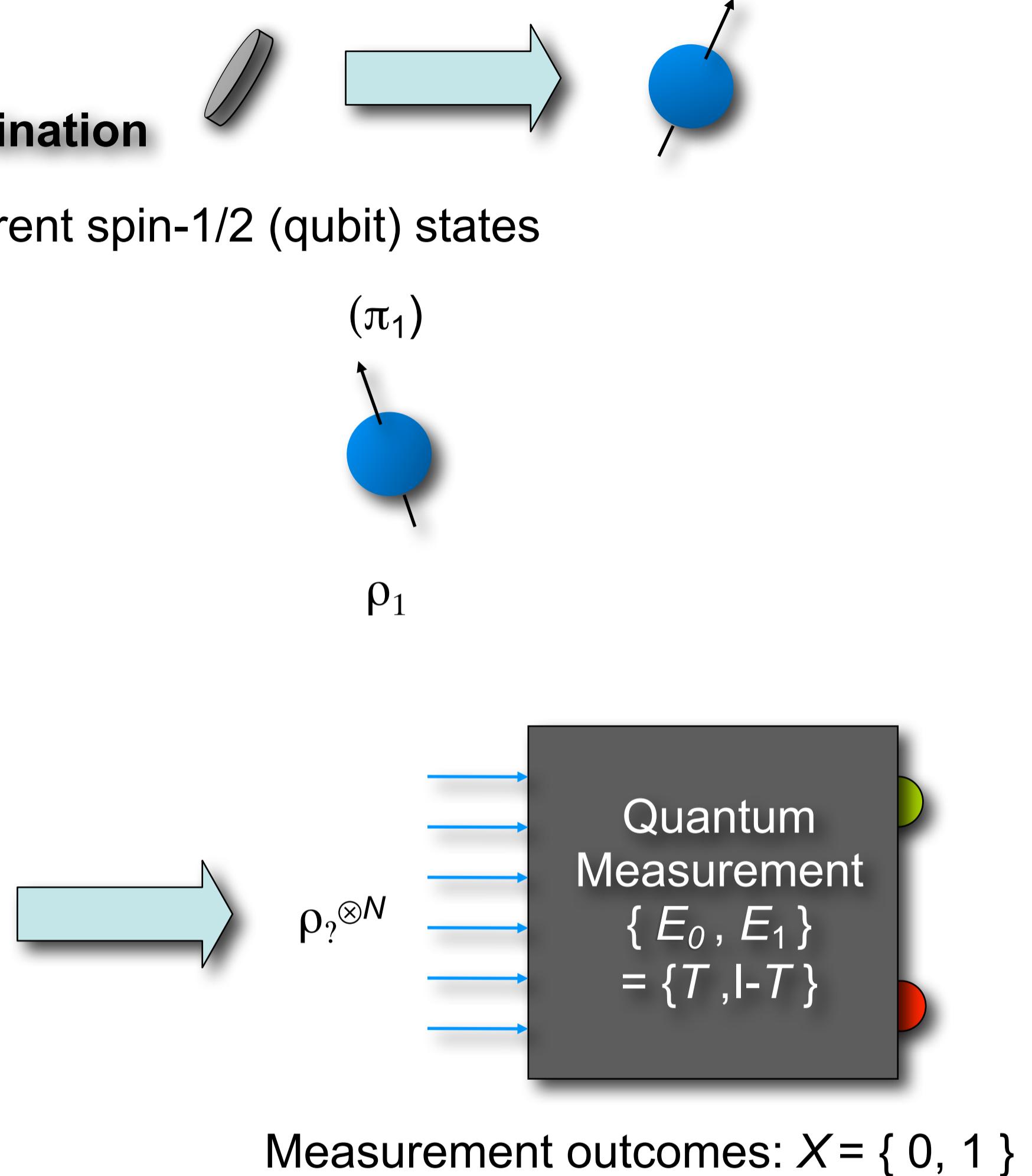
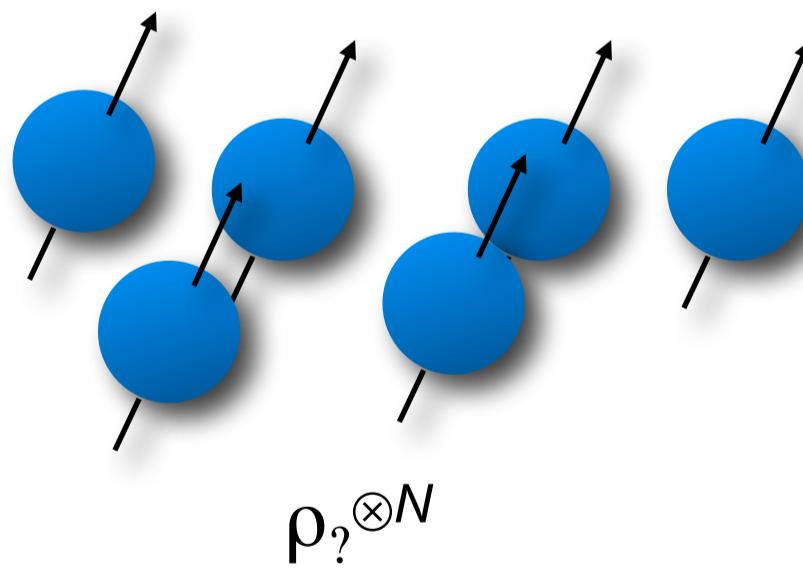
QUANTUM

Hypothesis testing/2-state discrimination

Simplest example: consider two different spin-1/2 (qubit) states



We are given N identical states $\rho_?$



Decide if we have been given ρ_0 or ρ_1 with minimal averaged error probability

$$P_e = \pi_0 \text{ prob}(1 | \rho_0) + \pi_1 \text{ prob}(0 | \rho_1)$$

QUANTUM

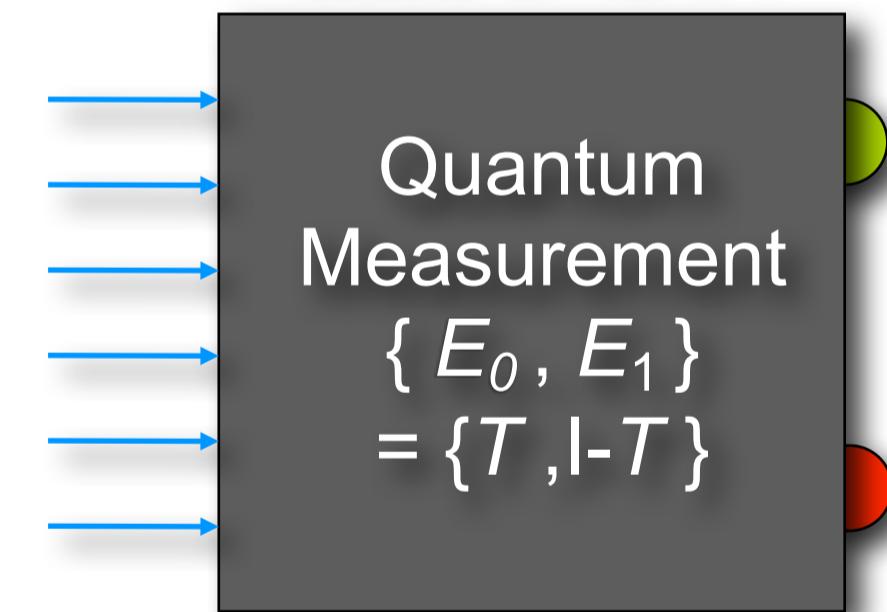
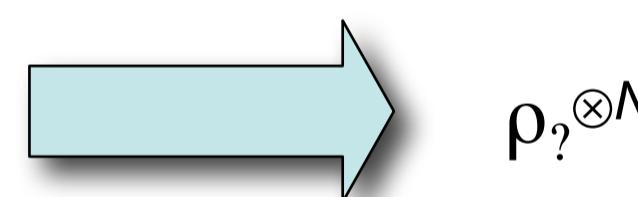
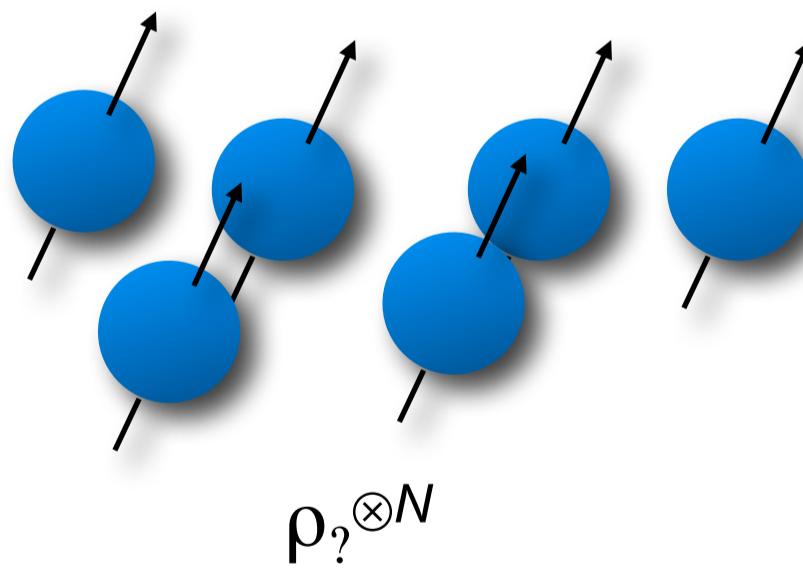
Hypothesis testing/2-state discrimination

Simplest example: consider two different spin-1/2 (qubit) states



What is the **best** measurement ?

We are given N identical states $\rho_?$



Measurement outcomes: $X = \{ 0, 1 \}$

Decide if we have been given ρ_0 or ρ_1 with minimal averaged error probability

$$P_e = \pi_0 \text{ prob}(1 | \rho_0) + \pi_1 \text{ prob}(0 | \rho_1)$$

2-state discrimination

QUANTUM

Hypothesis testing/2-state discrimination

QHT/2-SD are

- the most elementary problems in quantum information
- fundamental to quantum cryptography, quantum communication (channel coding), quantum computation, quantum engineering,...

QHT/2-SD are non-trivial because of

- no-cloning
- non-orthogonal (pure) states cannot be exactly distinguished
- mixed states with intersecting support cannot be exactly distinguished
- $[\rho_0, \rho_1] \neq 0$

Quantum 2-SD approaches:

- Unambiguous (No error is accepted) : Π_1, Π_2, Π_{inc}
 - Probability of inconclusive outcome*
- Minimum averaged error probability** (Bayesian)

The Quantum Chernoff Bound

K.M.R. Audenaert, J Calsamiglia, LI. Masanes, R. Munoz-Tapia, A. Acin,
E. Bagan and F. Verstraete
Phys. Rev. Lett. **98**, 160501 (2007)

PLAN OF THE TALK

- Introduction
- Single-copy 2-quantum state discrimination
- Multiple-copy *classical* 2-state discrimination ($[\rho_0, \rho_1] = 0$); *Class.* Chernoff bound
- Multiple-copy quantum 2-state discrimination by fixed local measurements
- Existing bounds on the asymptotic error probability P_e
- The Quantum Chernoff bound. Operational measure of distinguishability
- Induced metric on the space of states
- Summary, conclusions & overlook

QUANTUM

Two-state discrimination



C. W. Helstrom, *Quantum Detection and Estimation Theory*, Academic Press (New York), 1976.

A. S. Holevo, *Theor. Prob. Appl.* 23, 411 (1978).

Problem: consider **one** copy of two different states ρ_0, ρ_1

• Recall $P_e = \pi_0 \text{ prob}(1 | \rho_0) + \pi_1 \text{ prob}(0 | \rho_1)$



$$P_e = 1/2 (1 - \text{tr} |\Gamma|)$$

• Define the *Helstrom* matrix as $\Gamma = \pi_0 \rho_0 - \pi_1 \rho_1$

Proof:

$$\bullet \Gamma = \Gamma_+ - \Gamma_-; |\Gamma| = \Gamma_+ + \Gamma_- \Rightarrow 2\Gamma_+ = \Gamma + |\Gamma|$$

• Measurement operators are $\{ T, I - T \}$

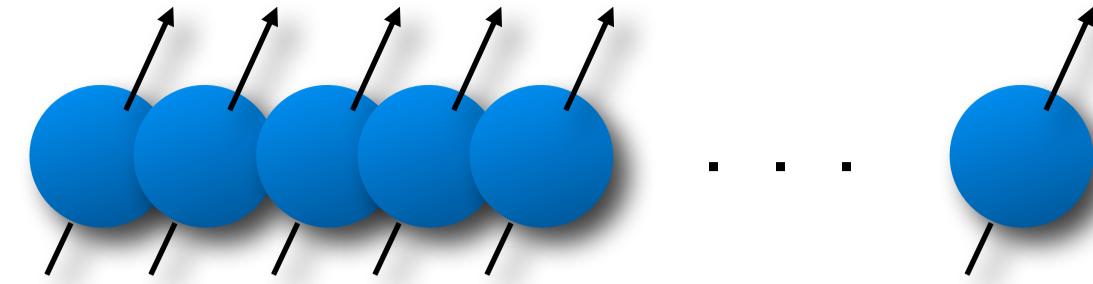
$$P_e = \pi_0 \text{ prob}(1 | \rho_0) + \pi_1 \text{ prob}(0 | \rho_1) = \pi_0 \text{ tr}[(I - T)\rho_0] + \pi_1 \text{ tr}(T\rho_1) = \pi_0 - \text{tr}(T\Gamma)$$

$$\geq \pi_0 - \text{tr} \Gamma_+ = \pi_0 - 1/2 (\pi_0 - \pi_1) - 1/2 \text{tr} |\Gamma|$$

• Bound attained when T is the projector over Γ_+ , which we denote as $\{\Gamma > 0\}$

QUANTUM

Multiple-copy two-state discrimination



Problem: consider N **copies** of two different states ρ_0, ρ_1

$$P_e = \frac{1}{2} (1 - \text{tr} |\Gamma|)$$

$$\text{where } \Gamma = \pi_0 \rho_0^{\otimes N} - \pi_1 \rho_1^{\otimes N}$$

The problem is formally solved

This expression does not say much about the asymptotic limit $N \rightarrow \infty$

Next, I show how the (classical) Chernoff bound can be derived from the expression above when $[\rho_0, \rho_1] = 0$

For simplicity, from now on $\pi_0 = \pi_1 = 1/2$

Then,

$$P_e = \frac{1}{2} (1 - 2 \text{tr} \Gamma_+)$$

QUANTUM → CLASSICAL

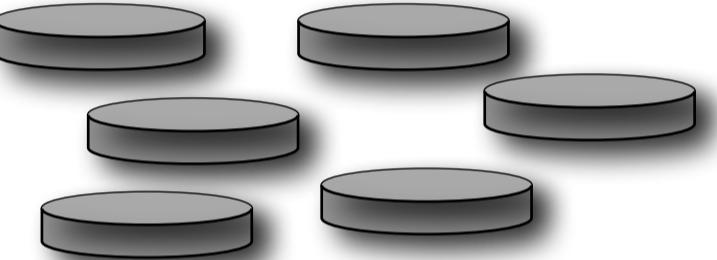
$$[\rho_0, \rho_1] = 0$$

Hypothesis testing/discrimination

$$H_0 \rightarrow \rho_0 = \begin{pmatrix} p_0 & 0 \\ 0 & 1-p_0 \end{pmatrix} \quad H_1 \rightarrow \rho_1 = \begin{pmatrix} p_1 & 0 \\ 0 & 1-p_1 \end{pmatrix}$$

$$X = \{ h, t \}$$

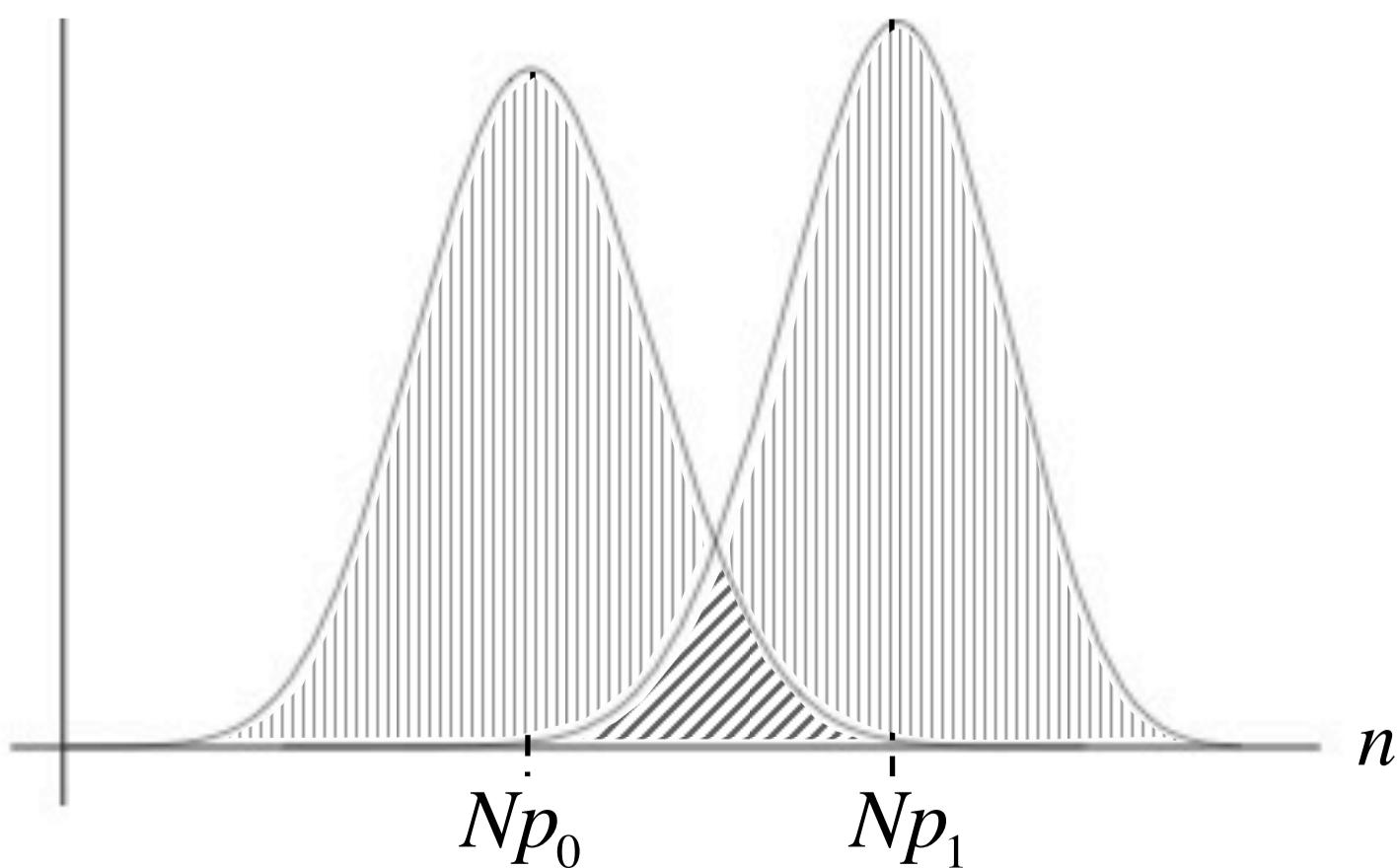
N identical coins

$$H_i \rightarrow \rho_i^{\otimes N} = \text{diag}\left[\{p_i^n(1-p_i)^{N-n}, \binom{N}{n}\}_{n=0}^N\right]$$


$$P_e = \frac{1}{2}(1 - \frac{1}{2} \text{tr} |\rho_0 - \rho_1|)$$

Assume

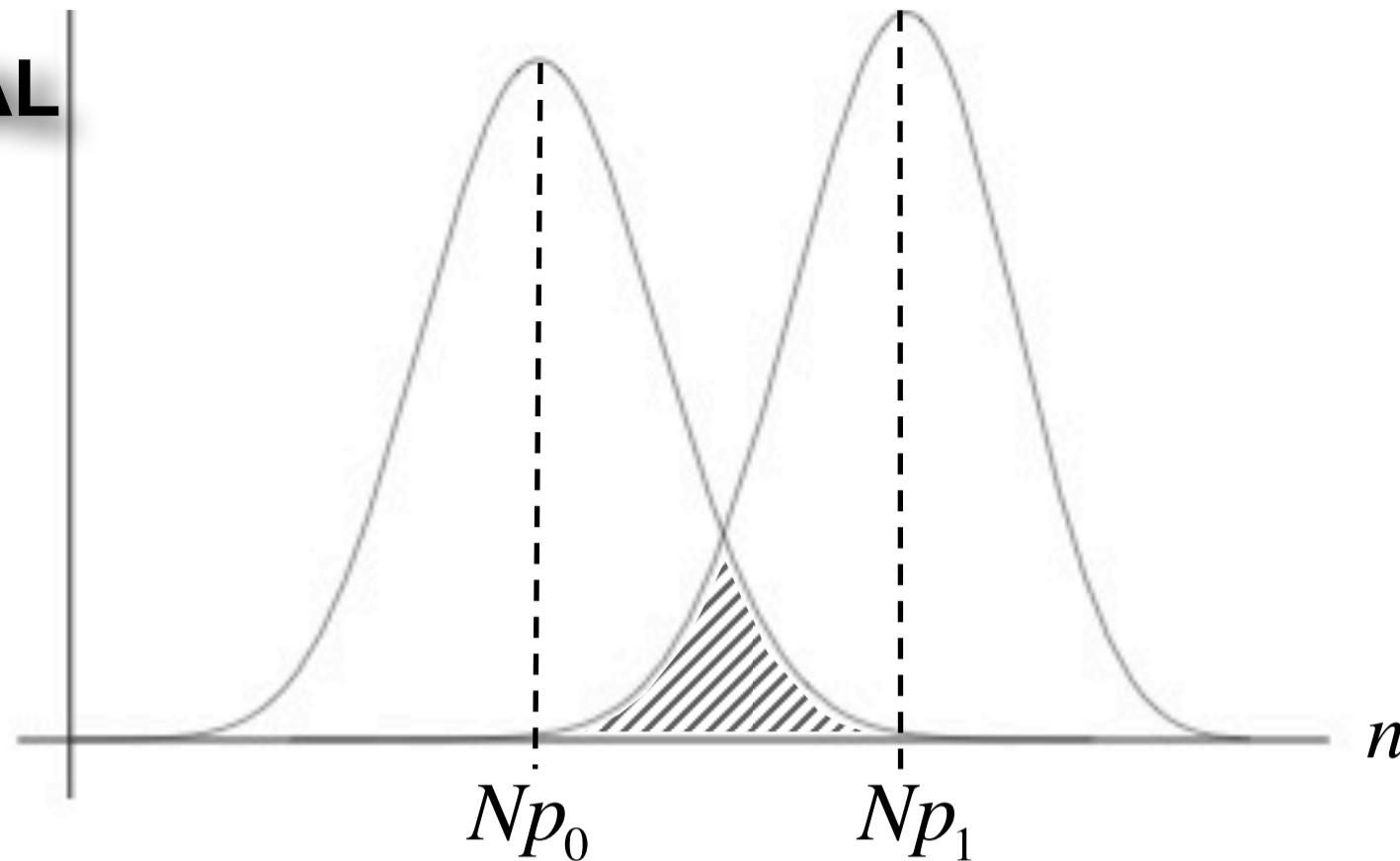
$$p_0 < p_1$$



QUANTUM → CLASSICAL

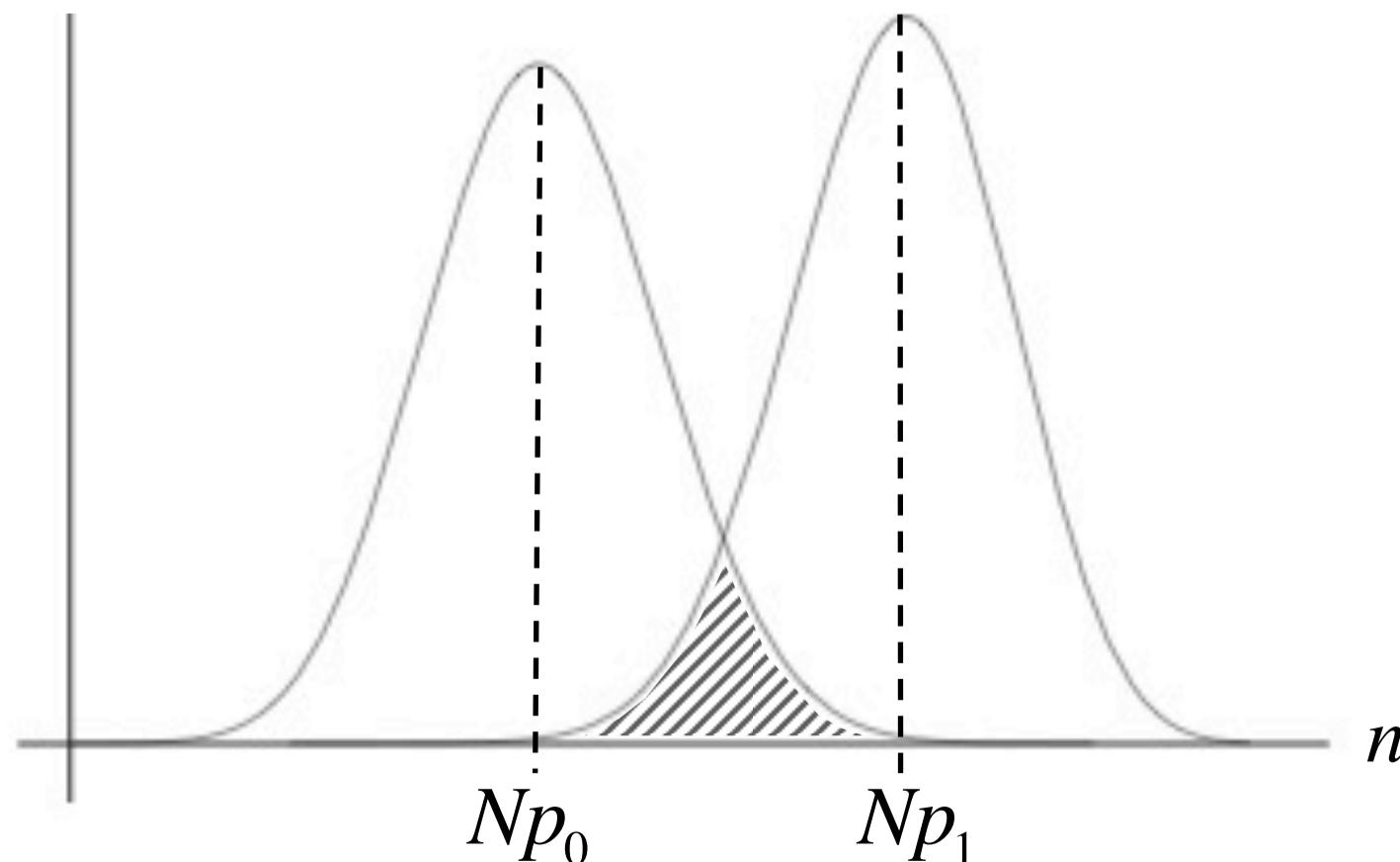
**Hypothesis testing
/discrimination**

$$[\rho_0, \rho_1] = 0$$



$$P_e = \frac{1}{2}(1 - \frac{1}{2} \text{tr} |\rho_0 - \rho_1|) = \frac{1}{2} \sum_{n=0}^N \binom{N}{n} \min \left\{ p_1^n (1-p_1)^{N-n}, p_0^n (1-p_0)^{N-n} \right\}$$

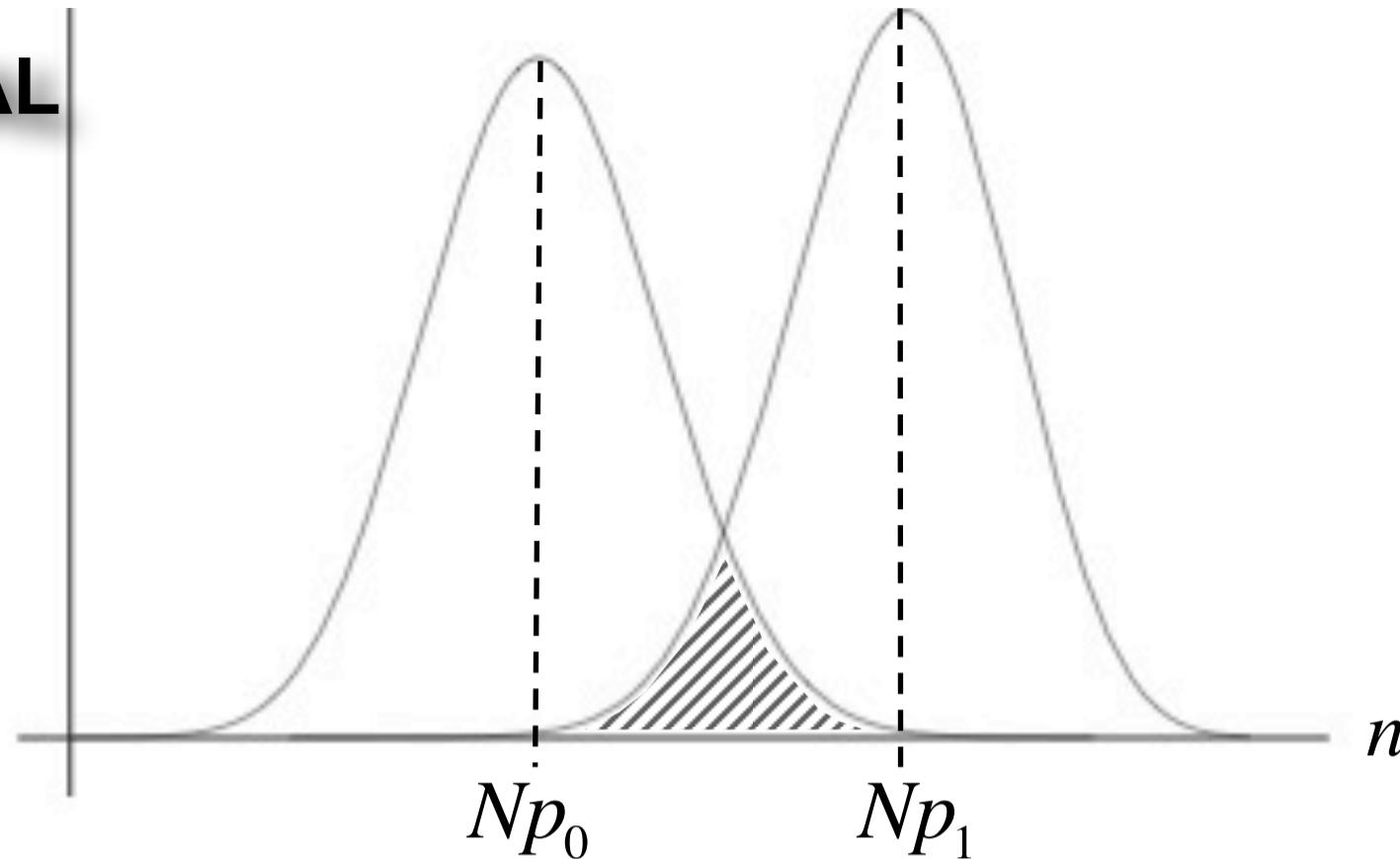
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QUANTUM → CLASSICAL

$$[\rho_0, \rho_1] = 0$$

**Hypothesis testing
/discrimination**



$$P_e = \frac{1}{2}(1 - \frac{1}{2} \text{tr} |\rho_0 - \rho_1|) = \frac{1}{2} \sum_{n=0}^N \binom{N}{n} \min \left\{ p_1^n (1-p_1)^{N-n}, p_0^n (1-p_0)^{N-n} \right\}$$

$$\min\{a,b\} \leq a^\lambda b^{1-\lambda}, \text{ for all } 0 \leq \lambda \leq 1$$

$$P_e \leq \frac{1}{2} \sum_{n=0}^N \binom{N}{n} (p_0^\lambda p_1^{1-\lambda})^n [(1-p_0)^\lambda (1-p_1)^{1-\lambda}]^{N-n} = \frac{1}{2} [p_0^\lambda p_1^{1-\lambda} + (1-p_0)^\lambda (1-p_1)^{1-\lambda}]^N$$

$$P_e \leq \frac{1}{2} \min_{0 \leq \lambda \leq 1} \sum_x P^{(N)}_0 \lambda(x) P^{(N)}_1 1-\lambda(x)$$

Chernoff bound

- General
- Achievable
- *Chernoff Information*

$$P_e \leq \frac{1}{2} 2^{-N C}$$

$$- \lim_{N \rightarrow \infty} \frac{1}{N} \log P_e \leq C$$

$$C = - \min_{0 \leq \lambda \leq 1} \log \sum_x P_0^\lambda(x) P_1^{1-\lambda}(x)$$

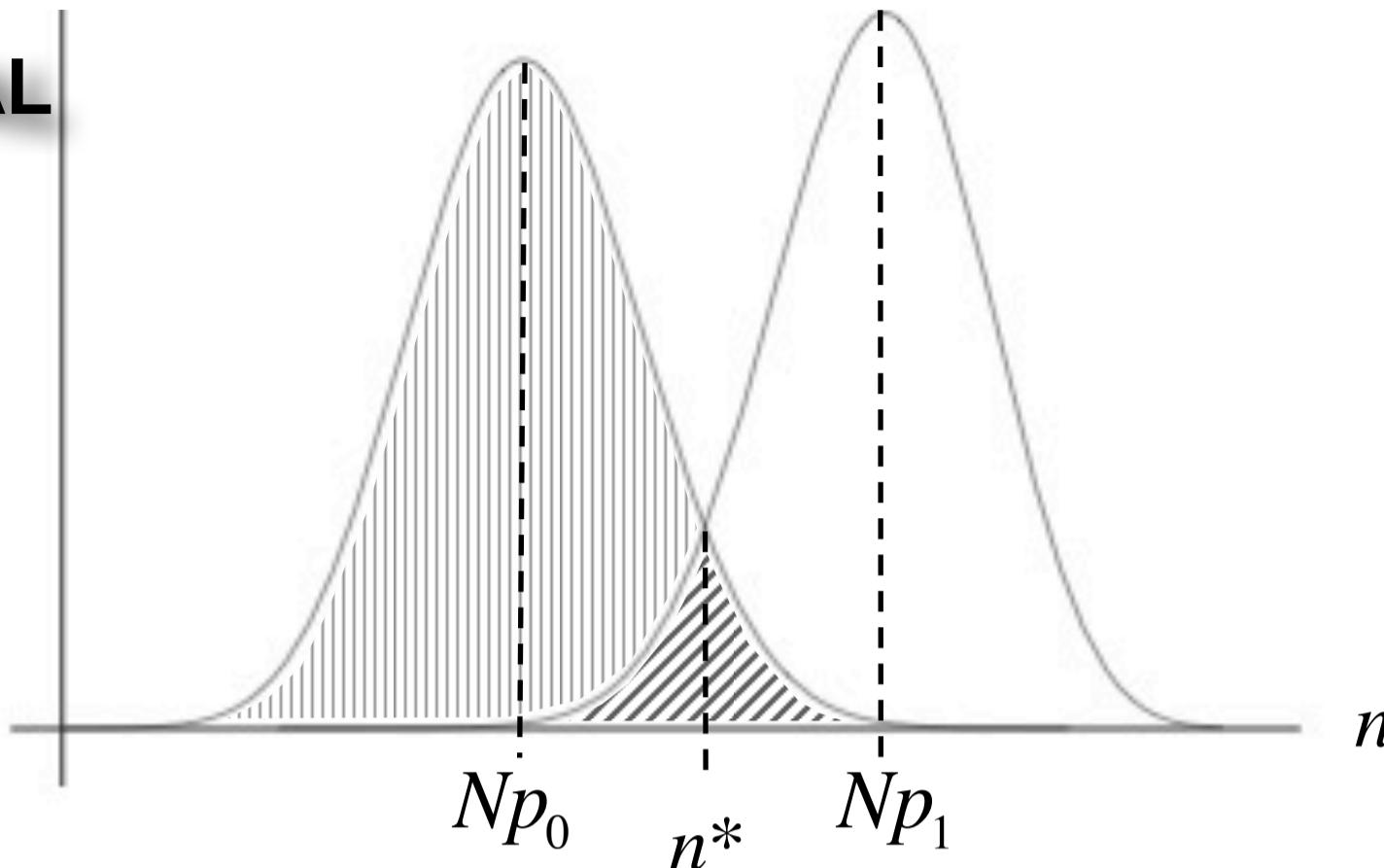
QUANTUM → CLASSICAL

$$[\rho_0, \rho_1] = 0$$

Hypothesis testing

To prove that Chernoff bound is **attainable** as $N \rightarrow \infty$

show it is also a **lower** bound to the error probability



$$P_e = \frac{1}{2} [1 - \text{tr}(\rho_0 - \rho_1)_+] \geq \frac{1}{2} \binom{N}{n^*} p_1^{n^*} (1-p_1)^{N-n^*}$$

Where n^* is defined by: $p_1^{n^*} (1-p_1)^{N-n^*} = p_0^{n^*} (1-p_0)^{N-n^*}$

The solution is:

$$\frac{n^*}{N} = \frac{\log \left[\frac{(1-p_0)}{(1-p_1)} \right]}{\log \left[\frac{p_1(1-p_0)}{p_0(1-p_1)} \right]} = p_{\lambda^*} = P_{\lambda^*}(h) \Rightarrow -\lim_{N \rightarrow \infty} \frac{1}{N} \log P_e \geq C$$

Best test: toss the N coins and count # of h
 • If $\#(h) < n^*$ accept H_0
 • If $\#(h) \geq n^*$ accept H_1

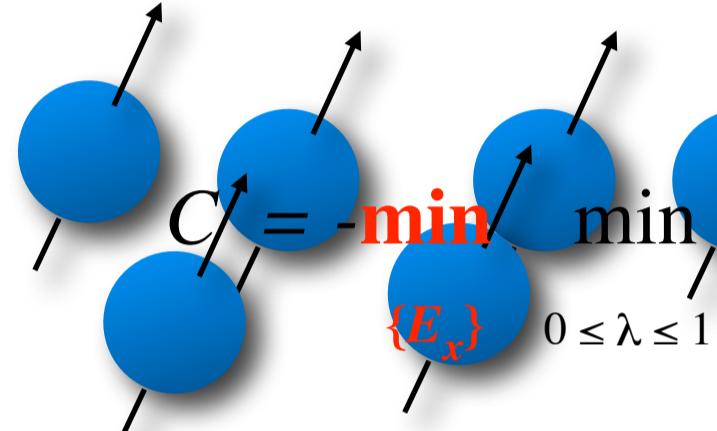
(measure σ_z on each copy)

Bayes' decision rule

Multiple-copy two state discrimination

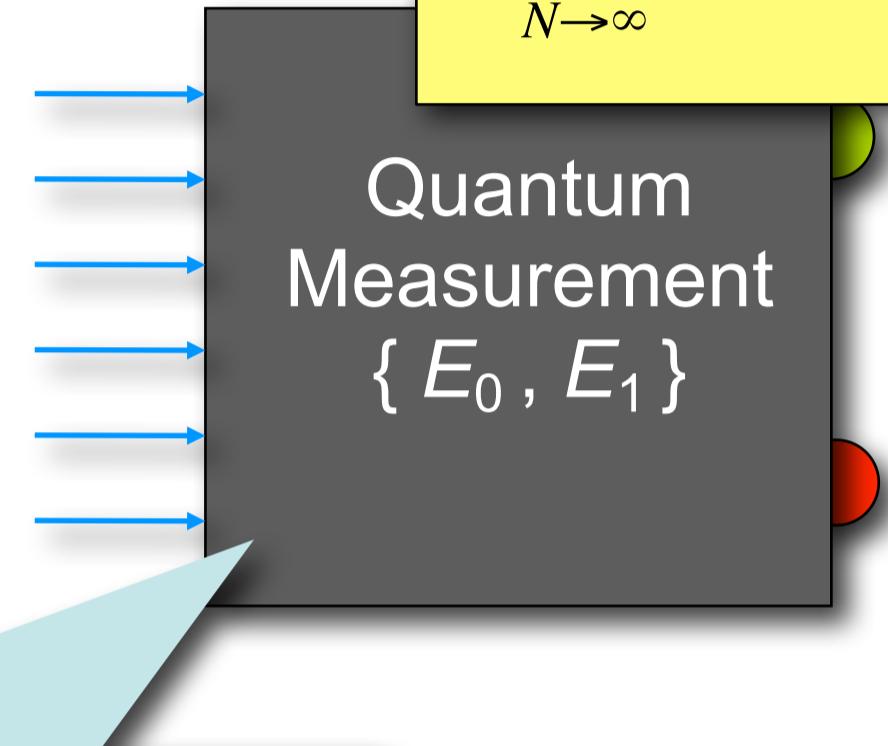
Long-standing problem.

Simplification: **fix local measurements**

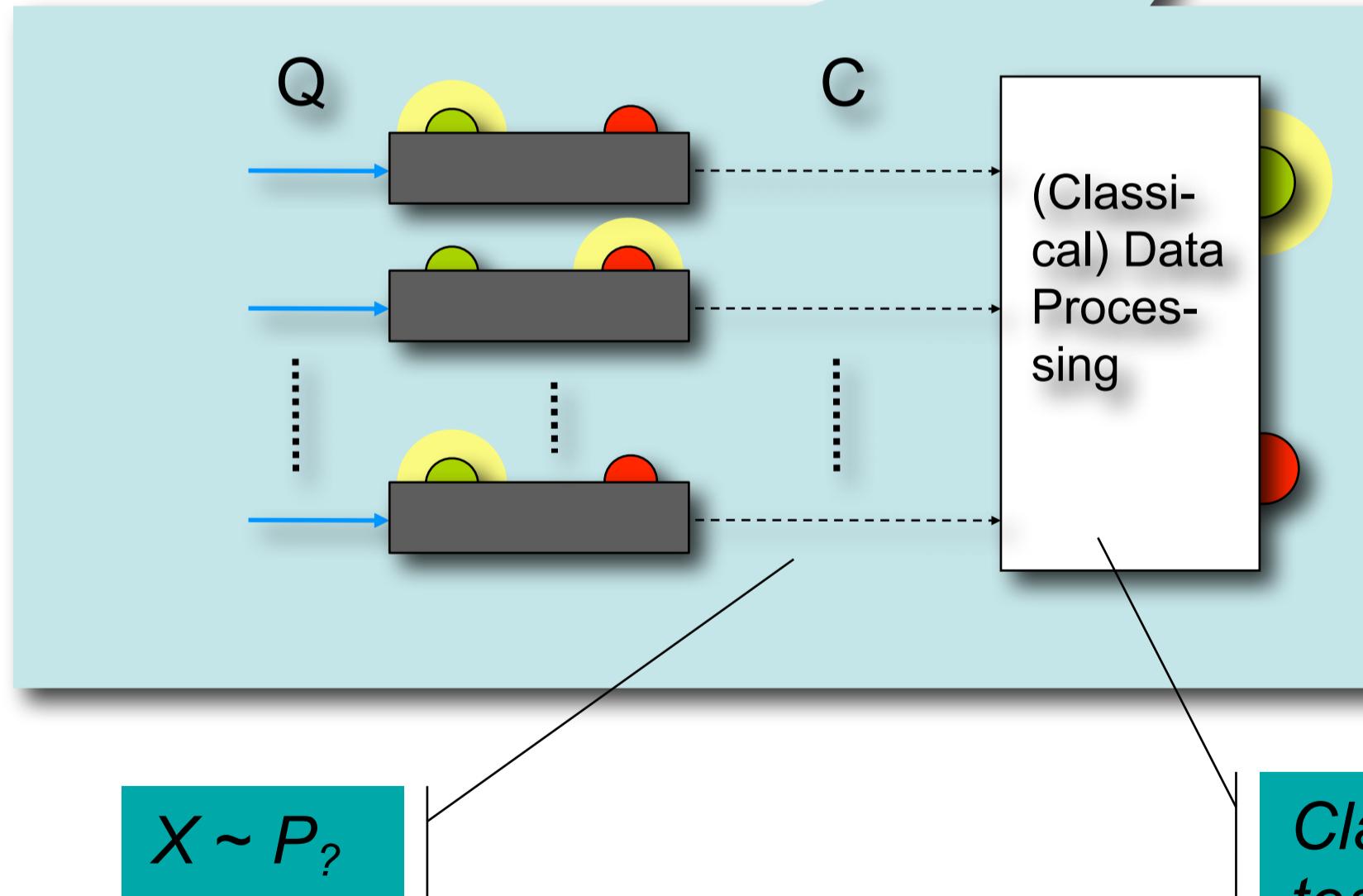


$$\rho_{?}^{\otimes N}$$

$$\log \sum_x P_0^{\lambda(E_x)} P_1^{1-\lambda(E_x)}$$



$$- \lim_{N \rightarrow \infty} \frac{1}{N} \log P_e \leq C$$



$$- \lim_{N \rightarrow \infty} \frac{1}{N} \log P_e \leq C$$

Multiple-copy two state discrimination

Long-standing problem.

Simplification: **fix local** measurements

FURTHER simplification (statistical overlap)

STILL FURTHER simplification: **qubits**

The fidelity lower bound **IS** the exponential rate **C** of decline for **fixed local** measurements, **unless** ρ_0 and ρ_1 (same purity) **are very pure**

$$- \lim_{N \rightarrow \infty} \frac{1}{N} \log P_e \leq C$$

$$C \geq -\frac{1}{2} \log[F(\rho_0, \rho_1)]$$

$$F(\rho_0, \rho_1) = [\text{tr} \sqrt{\sqrt{\rho_0} \rho_1 \sqrt{\rho_0}}]^2$$

where $F(\rho_0, \rho_1)$ is the **fidelity**, defined as:

$$F(\rho_0, \rho_1) = [\text{tr} \sqrt{\sqrt{\rho_0} \rho_1 \sqrt{\rho_0}}]^2$$

$$= |\langle \psi_0 | \psi_1 \rangle|^2$$

$$\left[\text{tr} \sqrt{|\psi_0\rangle\langle\psi_0| \rho_1 |\psi_0\rangle\langle\psi_0|} \right]^2 = \langle \psi_0 | \rho_1 | \psi_0 \rangle \left[\text{tr} \sqrt{|\psi_0\rangle\langle\psi_0|} \right]^2$$

$$= \langle \psi_0 | \rho_1 | \psi_0 \rangle [\text{tr} |\psi_0\rangle\langle\psi_0|]^2 = \langle \psi_0 | \rho_1 | \psi_0 \rangle$$

Multiple-copy two state discrimination

Long-standing problem.

Simplification: **fix local** measurements

FURTHER simplification (statistical overlap)

STILL FURTHER simplification: **qubits**

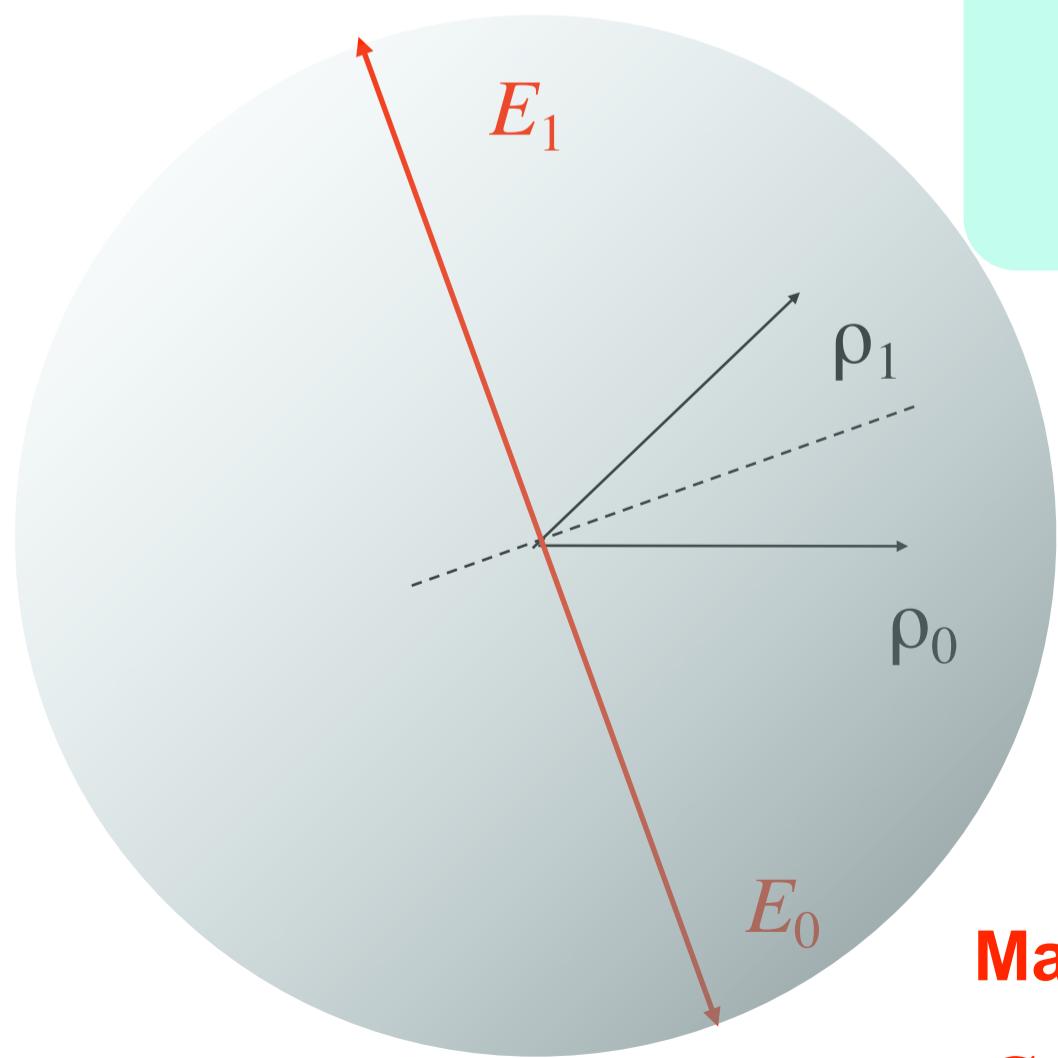
The fidelity lower bound **IS** the exponential rate **C** of decline for **fixed local** measurements, **unless** ρ_0 and ρ_1 (same purity) **are very pure**

$$- \lim_{N \rightarrow \infty} \frac{1}{N} \log P_e \leq C$$

$$C \geq -\frac{1}{2} \log[F(\rho_0, \rho_1)]$$

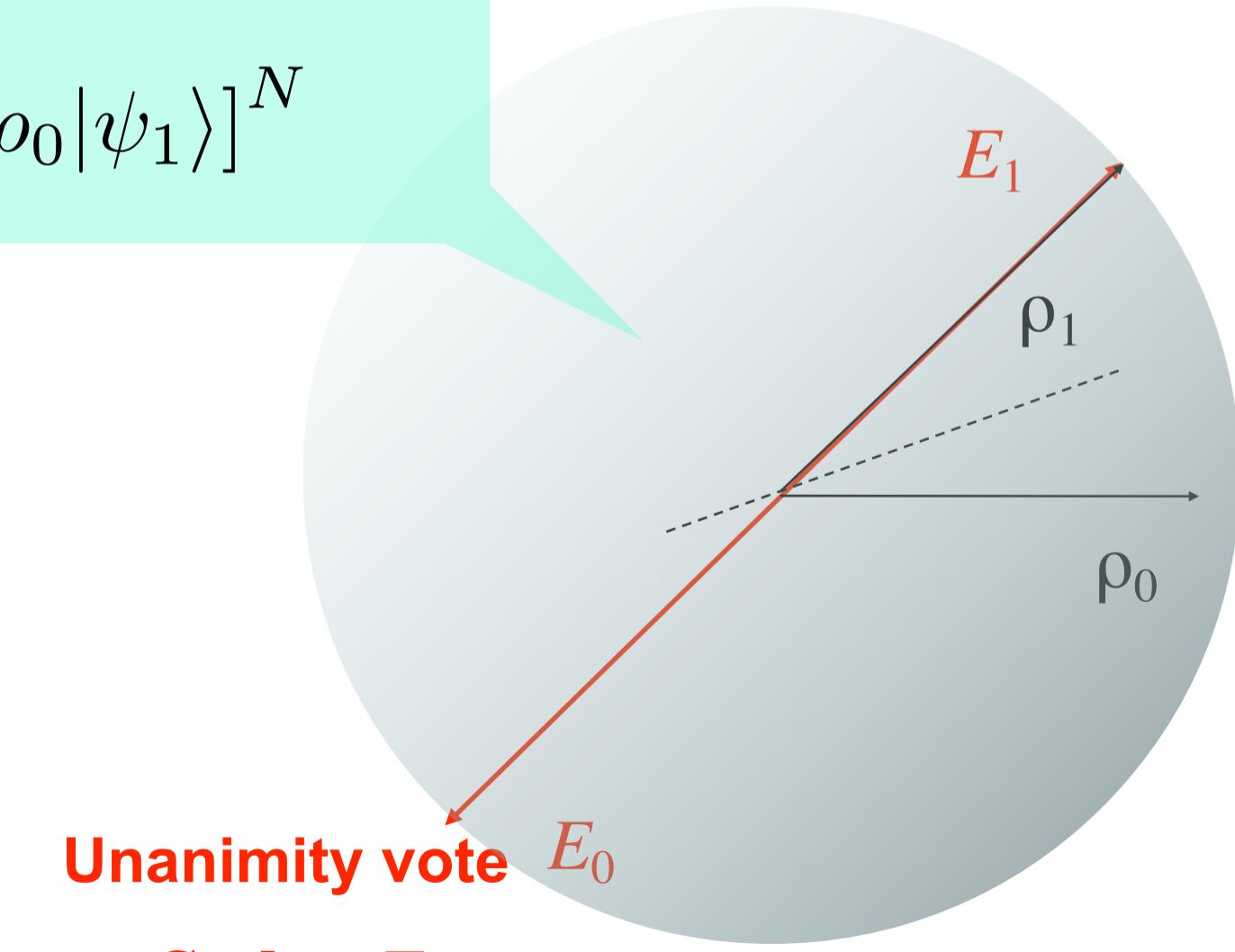
$$F(\rho_0, \rho_1) = [\text{tr} \sqrt{\sqrt{\rho_0} \rho_1 \sqrt{\rho_0}}]^2$$

$$\begin{aligned} P_e &= \frac{1}{2} [\text{tr} (\rho_0 |\psi_1\rangle\langle\psi_1|)]^N \\ &= \frac{1}{2} [\langle\psi_1|\rho_0|\psi_1\rangle]^N \end{aligned}$$



Majority vote

$$C = -\frac{1}{2} \log F$$



Unanimity vote

$$C = -\log F$$

Multiple-copy two state discrimination

Long-standing problem.

bounds to the trace distance (Fuchs & van de Graaf)

$$1 - \sqrt{F(\rho_0, \rho_1)} \leq \frac{1}{2} \text{tr} |\rho_0 - \rho_1| \leq \sqrt{1 - F(\rho_0, \rho_1)}$$

If one of the states is pure there is a tighter lower bound

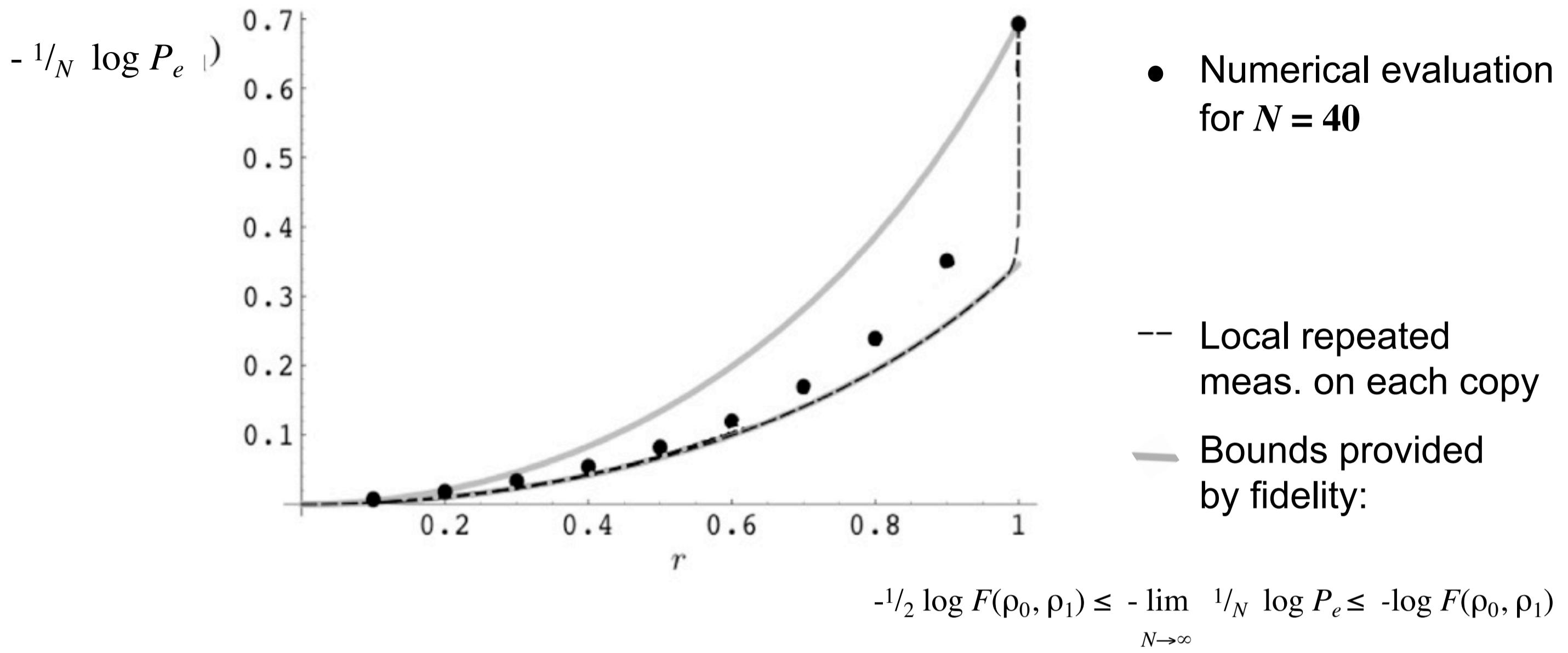
$$1 - F(\rho_0, \rho_1) \leq \frac{1}{2} \text{tr} |\rho_0 - \rho_1| \leq \sqrt{1 - F(\rho_0, \rho_1)}$$

It follows that

$$-\frac{1}{2} \log F(\rho_0, \rho_1) \leq -\lim_{N \rightarrow \infty} \frac{1}{N} \log P_e \leq -\log F(\rho_0, \rho_1)$$

If one of the states is pure

$$-\lim_{N \rightarrow \infty} \frac{1}{N} \log P_e = -\log F(\rho_0, \rho_1)$$



Ogawa, Nagaoka & Hayashi bounds

$$-\min\{\Psi(\rho_0\|\rho_1), \Psi(\rho_1\|\rho_0)\} \leq -\lim_{N \rightarrow \infty} \frac{1}{N} \log P_e \leq -\max\{D(\rho_0\|\rho_1), D(\rho_1\|\rho_0)\}$$

Where

$$D(\rho_0\|\rho_1) = -\text{tr} [\rho_0 (\log \rho_0 - \log \rho_0)]$$

is the quantum relative entropy and

$$\Psi(\rho_0\|\rho_1) = \min_{0 \leq \lambda \leq 1} \log \text{tr} [\rho_0 \rho_1^{\lambda/2} \rho_0^{-\lambda} \rho_1^{\lambda/2}]$$

Non of them are achievable!

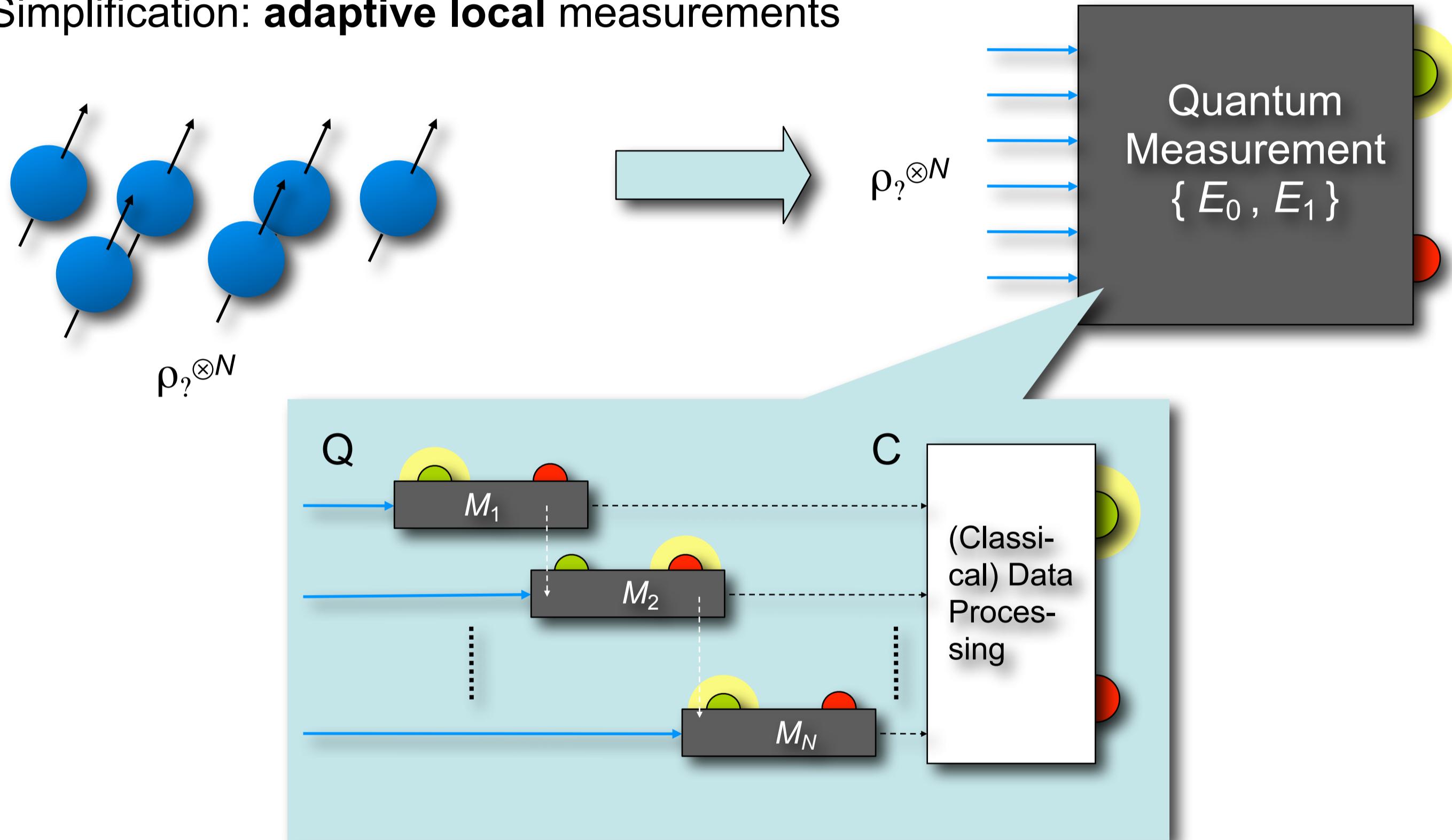
QUANTUM

Multiple-copy two state discrimination

$$[\rho_0, \rho_1] \neq 0$$

Long-standing problem.

Simplification: **adaptive local measurements**



For **pure states** this scheme (through Bayesian updating) is optimal **for any N**

J. Walgate et al., Phys. Rev. Lett. 85, 4972 (2000); S. Virmani, et al., Phys. Lett. A 228, 62 (2001).
 A. Acin, E. Bagan, M. Baig, Ll. Masanes & R. Munoz-Tapia, Phys. Rev. A 71 032338 (2005);

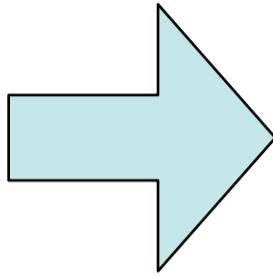
QUANTUM

Multiple-copy two state discrimination

$$[\rho_0, \rho_1] \neq 0$$

The quantum Chernoff bound

$$P_e \leq \frac{1}{2} \min_{0 \leq \lambda \leq 1} \sum_x P_0^\lambda(x) P_1^{1-\lambda}(x)$$



$$P_e \leq \frac{1}{2} \min_{0 \leq \lambda \leq 1} \text{tr} \rho_0^\lambda \rho_1^{1-\lambda}$$

naively expect

Show that, indeed, for $0 \leq \lambda \leq 1$ one has $P_e \leq \frac{1}{2} \text{tr} \rho_0^\lambda \rho_1^{1-\lambda}$

THE PROOF 1/5

$$[\rho_0, \rho_1] \neq 0$$

M. Hayashi's book

For $\lambda = 1/2$ choose $T = \{ \rho_0^{1/2} - \rho_1^{1/2} \geq 0 \}$

$$\begin{aligned} 2P_e &\leq \text{tr } \rho_0 (I-T) + \text{tr } \rho_1 T \\ &= \text{tr } \rho_0^{1/2} \rho_0^{1/2} \{ \rho_0^{1/2} - \rho_1^{1/2} < 0 \} + \text{tr } \rho_1^{1/2} \rho_1^{1/2} \{ \rho_0^{1/2} - \rho_1^{1/2} \geq 0 \} \\ &\leq \text{tr } \rho_0^{1/2} \rho_1^{1/2} \{ \rho_0^{1/2} - \rho_1^{1/2} < 0 \} + \text{tr } \rho_0^{1/2} \rho_1^{1/2} \{ \rho_0^{1/2} - \rho_1^{1/2} \geq 0 \} \\ &= \text{tr } \rho_0^{1/2} \rho_1^{1/2} (\{ \rho_0^{1/2} - \rho_1^{1/2} < 0 \} + \{ \rho_0^{1/2} - \rho_1^{1/2} \geq 0 \}) \\ &= \text{tr } \rho_0^{1/2} \rho_1^{1/2} \end{aligned}$$

where we have used

$$(\rho_1^{1/2} - \rho_0^{1/2}) \{ \rho_0^{1/2} - \rho_1^{1/2} < 0 \} \geq 0 \quad \text{and} \quad \{ \rho_0^{1/2} - \rho_1^{1/2} \geq 0 \} (\rho_0^{1/2} - \rho_1^{1/2}) \geq 0$$

THE PROOF 2/5

$$[\rho_0, \rho_1] \neq 0$$

For $0 \leq \lambda \leq 1/2$ choose $T = \{ \rho_0^{1-\lambda} - \rho_1^{1-\lambda} \geq 0 \}$

$$\begin{aligned} 2P_e &\leq \text{tr } \rho_0 (I-T) + \text{tr } \rho_1 T \\ &= \text{tr } \rho_0^{\lambda} \rho_0^{1-\lambda} \{ \rho_0^{1-\lambda} - \rho_1^{1-\lambda} < 0 \} + \text{tr } \rho_1^{1-\lambda} \rho_1^{\lambda} \{ \rho_0^{1-\lambda} - \rho_1^{1-\lambda} \geq 0 \} \\ &\leq \text{tr } \rho_0^{\lambda} \rho_1^{1-\lambda} \{ \rho_0^{1-\lambda} - \rho_1^{1-\lambda} < 0 \} + \text{tr } \rho_1^{1-\lambda} \rho_0^{\lambda} \{ \rho_0^{1-\lambda} - \rho_1^{1-\lambda} \geq 0 \} \\ &= \text{tr } \rho_0^{\lambda} \rho_1^{1-\lambda} (\{ \rho_0^{1-\lambda} - \rho_1^{1-\lambda} < 0 \} + \{ \rho_0^{\lambda} - \rho_1^{1-\lambda} \geq 0 \}) \\ &= \text{tr } \rho_0^{\lambda} \rho_1^{1-\lambda} \end{aligned}$$

where we have used

$$(\rho_1^{1-\lambda} - \rho_0^{1-\lambda}) \{ \rho_0^{1-\lambda} - \rho_1^{1-\lambda} < 0 \} \geq 0 \quad \text{and} \quad \text{tr } \rho_1^{1-\lambda} (\rho_0^{\lambda} - \rho_1^{\lambda}) \{ \rho_0^{1-\lambda} - \rho_1^{1-\lambda} \geq 0 \} \geq 0$$

Lemma: A, B positive operators and $0 \leq t \leq 1$

$$\text{tr } B (A^t - B^t) \{ A - B \geq 0 \} \geq 0$$

$$\begin{aligned} A &= \rho_0^{1-\lambda} \\ B &= \rho_1^{1-\lambda} \\ t &= \lambda/(1-\lambda) \end{aligned}$$

THE PROOF 3/5

$$[\rho_0, \rho_1] \neq 0$$

For $1/2 \leq \lambda \leq 1$ choose $T = \{ \rho_0^\lambda - \rho_1^\lambda > 0 \}$

$$\begin{aligned} 2P_e &\leq \text{tr } \rho_0(I-T) + \text{tr } \rho_1 T \\ &= \text{tr } \rho_0^\lambda \rho_0^{1-\lambda} \{ \rho_0^\lambda - \rho_1^\lambda \leq 0 \} + \text{tr } \rho_1^\lambda \rho_1^{1-\lambda} \{ \rho_0^\lambda - \rho_1^\lambda > 0 \} \\ &\leq \text{tr } \rho_0^\lambda \rho_1^{1-\lambda} \{ \rho_0^\lambda - \rho_1^\lambda \leq 0 \} + \text{tr } \rho_0^\lambda \rho_1^{1-\lambda} \{ \rho_0^\lambda - \rho_1^\lambda > 0 \} \\ &= \text{tr } \rho_0^\lambda \rho_1^{1-\lambda} (\{ \rho_0^\lambda - \rho_1^\lambda \leq 0 \} + \{ \rho_0^\lambda - \rho_1^\lambda > 0 \}) \\ &= \text{tr } \rho_0^\lambda \rho_1^{1-\lambda} \end{aligned}$$

where we have used

$$\{ \rho_0^\lambda - \rho_1^\lambda > 0 \} (\rho_0^\lambda - \rho_1^\lambda) \geq 0 \quad \text{and} \quad \text{tr } \rho_0^\lambda (\rho_1^{1-\lambda} - \rho_0^{1-\lambda}) \{ \rho_1^\lambda - \rho_0^\lambda \geq 0 \} \geq 0$$

Lemma: A, B positive operators and $0 \leq t \leq 1$

$$\text{tr } B (A^t - B^t) \{ A - B \geq 0 \} \geq 0$$

$$\begin{aligned} A &= \rho_1^\lambda \\ B &= \rho_0^\lambda \\ t &= (1-\lambda)/\lambda \end{aligned}$$

THE PROOF 4/5

Lemma: A, B positive operators and $0 \leq t \leq 1$

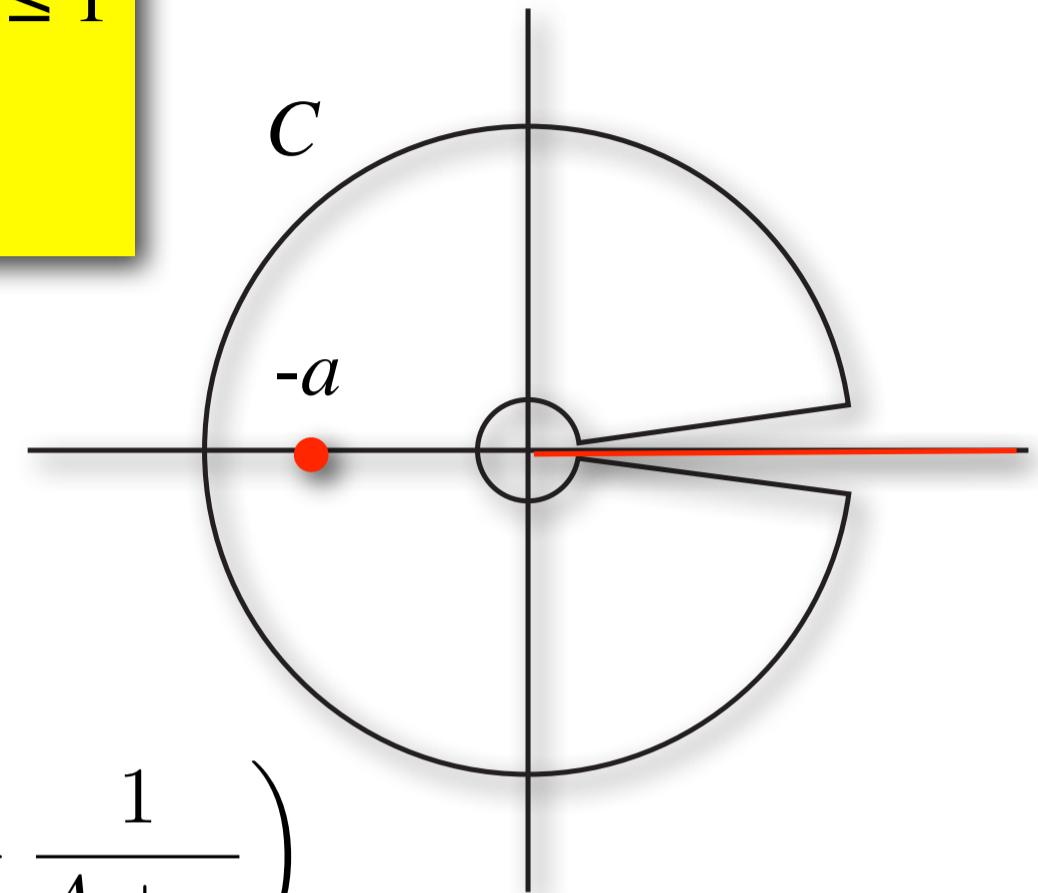
$$\operatorname{tr} B (A^t - B^t) \{ A - B \geq 0 \} \geq 0$$

Proof:

$$a^t = \frac{\sin(t\pi)}{\pi} \int_0^{+\infty} dx \frac{ax^{t-1}}{a+x},$$

for $a \geq 0$ and $0 \leq t \leq 1$

$$\oint_C dz \frac{z^{t-1}}{a+z}$$



$$A^t - B^t \rightarrow A(A+x)^{-1} - B(B+x)^{-1} = x \left(\frac{1}{B+x} - \frac{1}{A+x} \right)$$

$$\frac{1}{b} - \frac{1}{a} = \int_0^1 ds \frac{d}{ds} \frac{-1}{b + (a-b)s}$$

$$\frac{d}{ds} \frac{-1}{X} = \frac{1}{X} \frac{dX}{ds} \frac{1}{X}$$

$$A^t - B^t = \frac{\sin t\pi}{\pi} \int_0^\infty dx x^t \int_0^1 \frac{ds}{s} \frac{1}{B + (A-B)s + x} s(A-B) \frac{1}{B + (A-B)s + x}$$

Lemma: $V \geq 0$, Δ hermitian and $V^{-1} \geq \Delta + x$

$$\operatorname{tr} (V^{-1} - \Delta - x)(V \Delta V) \{ \Delta \geq 0 \} \geq 0$$

$$\begin{aligned} \Delta &= (A-B)s \\ V &= (B + \Delta + x)^{-1} \end{aligned}$$

THE PROOF 5/5

Lemma: $V \geq 0$, Δ hermitian and $V^{-1} \geq \Delta + x$

$$\operatorname{tr} (V^{-1} - \Delta - x)(V\Delta V) \{ \Delta \geq 0 \} \geq 0$$

$$V = V V^{-1} V \geq V(\Delta + x) V = V \Delta V + x V^2$$

Basis where $\Delta = \Delta_+ - \Delta_-$ is diagonal

$$\Delta = \begin{pmatrix} \Delta_+ & 0 \\ 0 & -\Delta_- \end{pmatrix}$$

$$V = \begin{pmatrix} V_{++} & V_{+-} \\ V_{-+} & V_{--} \end{pmatrix}$$

$$\begin{aligned} \operatorname{tr} (V^{-1} - \Delta - x)(V\Delta V) \{ \Delta \geq 0 \} &= \operatorname{tr} [\Delta V - (\Delta + x)(V\Delta V)] \{ \Delta \geq 0 \} \\ &= \operatorname{tr} [\Delta_+ V_{++} - (\Delta_+ + x)(V\Delta V)_{++}] \\ &\geq \operatorname{tr} [\Delta_+ (V\Delta V)_{++} + x \Delta_+(V^2)_{++} - (\Delta_+ + x)(V\Delta V)_{++}] \\ &= x \operatorname{tr} [\Delta_+(V^2)_{++} - (V\Delta V)_{++}] \\ &= x \operatorname{tr} [\Delta_+(V_{++})^2 + \Delta_+ V_{+-} V_{-+} - (V_{++} \Delta_+ V_{++} - V_{+-} \Delta_- V_{-+})] \\ &= x \operatorname{tr} [\Delta_+ V_{+-} V_{-+} + \Delta_- V_{-+} V_{+-}] \\ &= x \operatorname{tr} [\Delta_+ V_{+-} (V_{+-})^\dagger + \Delta_- V_{-+} (V_{-+})^\dagger] \geq 0 \end{aligned}$$



THE SOLUTION

$$[\rho_0, \rho_1] \neq 0$$

K.M.R. Audenaert, J Calsamiglia, LI. Masanes, R. Munoz-Tapia, A. Acin, E. Bagan and F. Verstraete, quant-ph/0610027

Theorem: Let A and B be two positive semi-definite operators. Then, for all $0 \leq \lambda \leq 1$ one has

$$\text{tr}|A-B| + 2 \text{tr} A^\lambda B^{1-\lambda} - \text{tr} A - \text{tr} B \geq 0$$

It follows that for any two density matrices one has

$$P_e \leq \frac{1}{2} \min_{0 \leq \lambda \leq 1} \text{tr} \rho_0^\lambda \rho_1^{1-\lambda}$$

Looks like naive generalization of CB prob $\rightarrow \rho$

For N -copy discrimination: $\text{tr} (\rho_0^{\otimes N})^\lambda (\rho_1^{\otimes N})^{1-\lambda} = (\text{tr} \rho_0^\lambda \rho_1^{1-\lambda})^N$

$$P_e \leq \frac{1}{2} 2^{-NQ}$$

$$- \lim_{N \rightarrow \infty} \frac{1}{N} \log P_e \leq Q$$

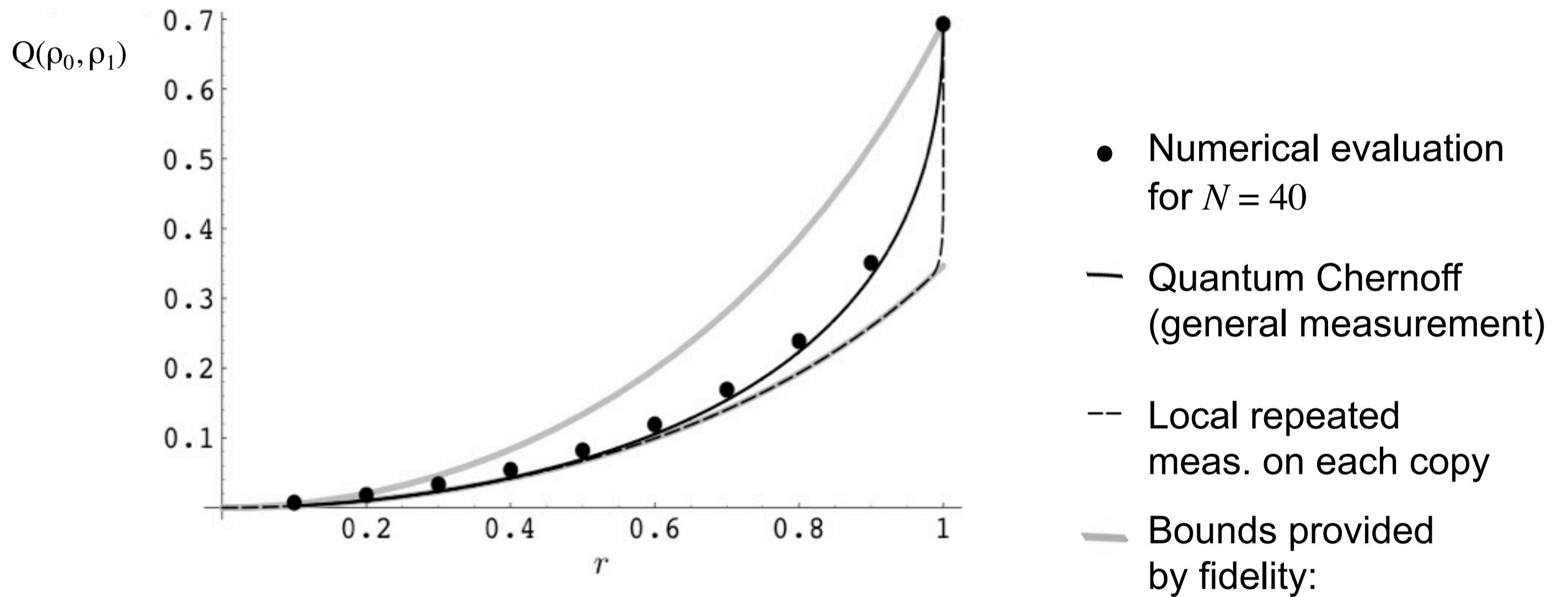
$$Q = - \min_{0 \leq \lambda \leq 1} \log \text{tr} \rho_0^\lambda \rho_1^{1-\lambda}$$

- General
- Achievable
- **Quantum Chernoff Information**; great distinguishability measure

Attainability follows from lower bound $Q \leq - \lim_{N \rightarrow \infty} \frac{1}{N} \log P_e$

M. Nussbaum & A. Szkola,
quant-ph/0607216

- Two qubit states with
- Bloch vectors have relative angle $\theta = \pi / 2$
 - Equal purity $r = r_1 = r_2$



$$-\frac{1}{2} \log F(\rho_0, \rho_1) \leq -\lim_{N \rightarrow 0} \frac{1}{N} \log P_e \leq -\log F(\rho_0, \rho_1)$$

INDUCED METRIC ON THE SPACE OF STATES

- $Q(\rho_1, \rho_2)$ provides an **operational** distinguishability measure, which can be **easily computed**. **Induces a natural metric ds_Q^2 in the space of states:**
- The larger $Q(\rho_1, \rho_2)$ & ds^2 the more distinguishable the states are

$$ds_Q^2 = Q[\rho(\theta), \rho(\theta + d\theta)] = d\theta^t g d\theta$$

$$ds_Q^2 = \frac{1}{2} \sum_{ij} \frac{|\langle i | d\rho | j \rangle|^2}{(\sqrt{\lambda_i} + \sqrt{\lambda_j})^2}$$

where $\rho = \sum_i \lambda_i | i \rangle \langle i |$

For qubits

$$ds_Q^2 = \frac{1}{8} \left[\frac{dr^2}{1-r^2} + 2(1-\sqrt{1-r^2})d\Omega^2 \right]$$

Compare with the metric induced by **classical Chernoff** for local repeated measurements

$$(r < 1) \quad ds_B^2 = \frac{1}{8} \left[\frac{dr^2}{1-r^2} + r^2 d\Omega^2 \right]$$

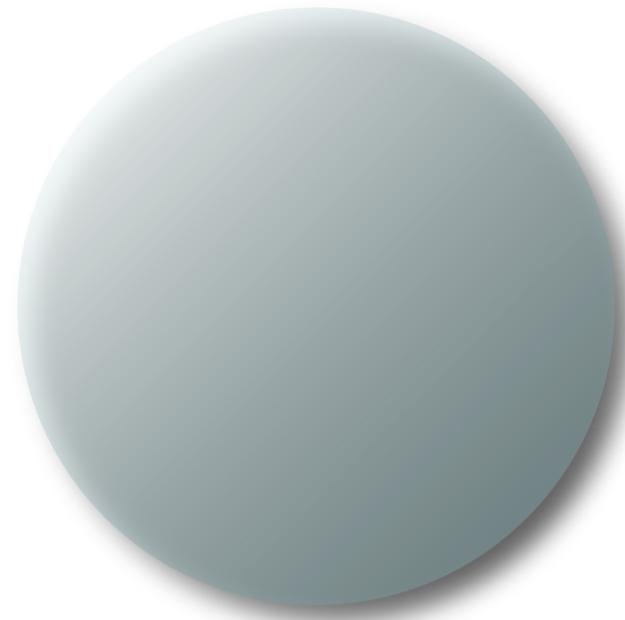
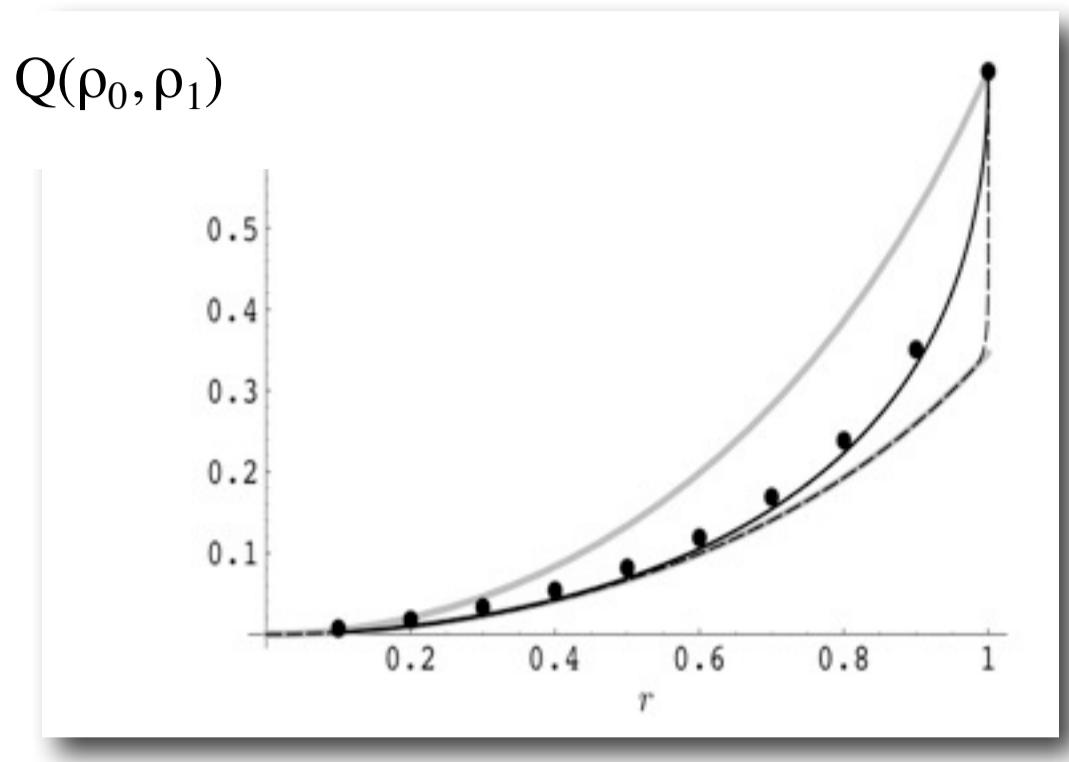
Bures metric

$$ds_Q^2 \geq ds_B^2 \quad (r = 1)$$

$$ds_{FS}^2 = \frac{1}{4} d\Omega^2$$

Fubini-Study metric

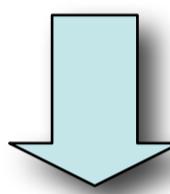
All the metrics in this slide are monotone metrics: **distinguishability & ds^2 do not increase under trace-preserving positive maps**



Embed the Bloch sphere in \mathbf{R}^4
(Uhlmann)

Q. Chernoff

$\frac{1}{2} \mathbf{I}$



Bures

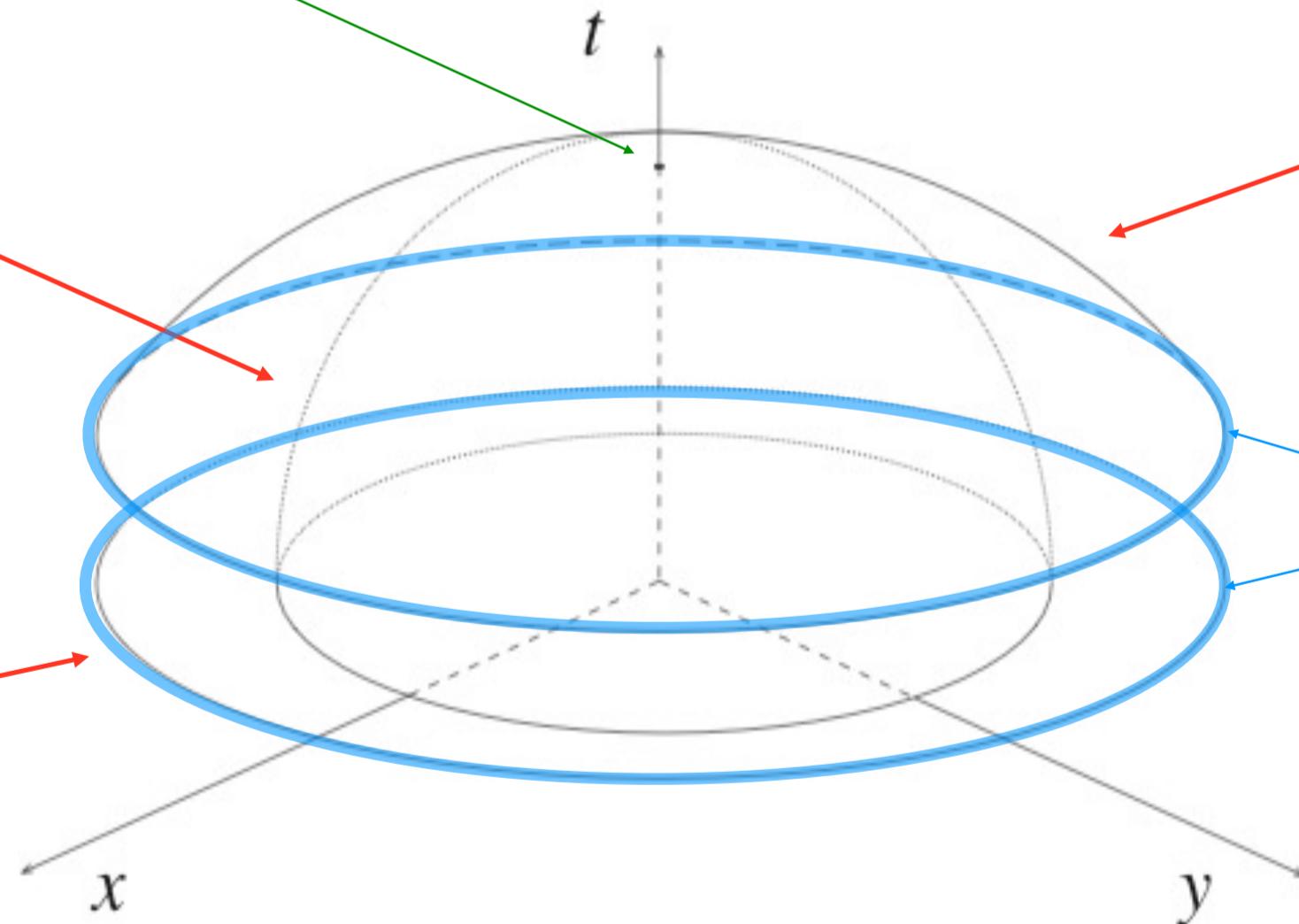
$$ds_B^2 = \frac{1}{8} \left[\frac{dr^2}{1-r^2} + r^2 d\Omega^2 \right]$$

$$ds_Q^2 = \frac{1}{8} \left[\frac{dr^2}{1-r^2} + 2 \left(1 - \sqrt{1-r^2} \right) d\Omega^2 \right]$$

Pure states

Fubini-Study

$$ds_{FS}^2 = \frac{1}{4} d\Omega^2$$



SUMMARY, CONCLUSIONS, ...

- I have **presented** and have **given the solution** to a long-standing **fundamental** open problem
 - Presented **asymptotic (Chernoff-type) bound** to P_e
 - Discussed approximations & loose upper and lower bounds to P_e
 - Standard (classical statistics) approach based on repeated local measurements + (classical) Chernoff bound;
 - **Fidelity bounds** and ONH bounds
 - Shown a powerful lemma + bound to the trace distance of positive operators
 - Distinguishability measure & **metric** on space of states (geometry)
 - Interpolates Bures & Fubini-Study.
 - Monotone/Contractive (under positive maps)
- Open questions
 - Can $Q(\rho_0, \rho_1)$ be also attained by LOCC/separable measurements or is it fully “quantum”?
 - Exploit applicability of $Q(\rho_0, \rho_1)$ / lemma / bound on trace norm
 - *Error Exponent in Asymmetric Quantum Hypothesis Testing and Its Application to Classical-Quantum Channel coding.* M. Hayashi, quant-ph/0611013
 - *The Converse Part of Theorem for Quantum Hoeffding Bound,* H. Nagaoka, quant-ph/0611289

Thanks for your attention